Mathematical Analysis and Stochastic Stability of Nonlinear Epidemic Model with Incidence Rate

Abstract

In this work, we consider a nonlinear epidemic model with temporary immunity and saturated incidence rate. Size N(t) at time t, is divided into three sub classes, with N(t)=S(t)+I(t)+Q(t); where S(t), I(t) and Q(t) denote the sizes of the population susceptible to disease, infectious and quarantine members with the possibility of infection through temporary immunity, respectively.

We have made the following contributions:

1. The local stabilities of the infection-free equilibrium and endemic equilibrium are; analyzed, respectively. The stability of a disease-free equilibrium and the existence of other nontrivial equilibria can be determine by the ratio called the basic reproductive number,
2. This paper study the reduce model with replace S with N, which does not have non-trivial periodic orbits with conditions.
3. The endemic -disease point is globally asymptotically stable if $R_{0} > 1$; and study some proprieties of equilibrium with theorems under some conditions.
4. Finally the stochastic stabilities with the proof of some theorems.

In this work, we have used the different references cited in different studies and especially the writing of the non-linear epidemic mathematical model with [1-7]. We have used the other references for the study the different stability and other sections with [8-26]; and sometimes the previous references.

Keywords: Local and global stability; stochastic stability; incidence rate; nonlinear epidemic model.

1 Introduction

This paper considers the following epidemic model with temporary immunity and saturated incidence rate.

$$\begin{cases} \dot{S}(t) = \rho + \lambda - \nu - (\mu + d)\mathrm{S}(\mathrm{t}) - \dfrac{(\beta + k)\mathrm{S}(\mathrm{t})\mathrm{I}(\mathrm{t})}{1 + bI(\mathrm{t})} + \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau), \\ \dot{I}(t) = \dfrac{(\beta + k)\mathrm{S}(\mathrm{t})\mathrm{I}(\mathrm{t})}{1 + bI(\mathrm{t})} - (\mu_1 + d)I(\mathrm{t}) - \gamma I(\mathrm{t}), \\ \dot{Q}(t) = \gamma I(\mathrm{t}) - \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau) - (\mu_2 + d)Q(\mathrm{t}). \end{cases} \tag{1}$$

Consider a population of size N (t) at time t, this population is divide into three subclasses.

With N (t) = S (t) + I(t)+Q(t). Where S (t), I (t), and Q (t) denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively. The positive constants μ, μ_1, and μ_2 represent the death rates of susceptible, infectious and quarantine. Biologically, it is natural to assume that $\mu \leq \min\{\mu_1, \mu_2\}$. The positive constant d is natural mortality rate.

The positive constants ρ represent rate of incidence. The positive constant k is the rate of unknown members infected, which is detect by the system.

The positive constant γ represent the recovery rate of infection. The positive constant β is the average numbers of contacts infective for S and I. The positive constant ν is the parameter of emigration. The positive constant λ is the parameter of immigration.

The term $\gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau)$ reflects the fact that an individual has recovered from infection and still are alive after infectious period τ, where τ is the length of immunity period and, b is saturation constant.

The initial condition of (1) is givens as:

$$S(\eta) = \Phi_1(\eta),\ I(\eta) = \Phi_2(\eta), Q(\eta) = \Phi_3(\eta), -\tau \leq \eta \leq 0. \tag{2}$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in \mathbb{C}$ such that:

$$S(\eta) = \Phi_1(\eta) = \Phi_1(0) \geq 0,\ I(\eta) = \Phi_2(\eta) = \Phi_2(0) \geq 0, Q(\eta) = \Phi_3(\eta) = \Phi_3(0) \geq 0.$$

Let $\mathbb{C}$ denote the Banach space $\mathbb{C}$ ([-τ, 0], $\mathbb{R}^3$) of continuous functions mapping the interval

[-τ, 0] into $\mathbb{R}^3$. With a biological meaning, we further assume that

$\Phi_i(\eta) = \Phi_i(0) \geq 0$, for i = 1, 2, 3.

Hence, system (1) rewritten as:

$$\begin{cases} \dot{S}(t) = \rho + \lambda - \nu - (\mu + d)\mathrm{S} - \dfrac{(\beta + k)\mathrm{SI}}{1 + bI} + \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau), \\ \dot{I}(t) = \dfrac{(\beta + k)\mathrm{SI}}{1 + bI} - (\mu_1 + d + \gamma)I\ , \\ \dot{Q}(t) = \gamma I - \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau) - (\mu_2 + d)Q\,. \end{cases} \tag{3}$$

With the initial conditions (2) where,

$$\Phi_i(0) \geq 0, -\tau \leq \eta \leq 0 \text{ , for i =1, 2, 3.} \tag{4}$$

The region $\Omega = \left\{ (S, I, Q) \in \mathbb{R}^3_+, S + I + Q \leq N < \frac{\rho + \lambda - \nu}{\mu + d} \right\}$ is positively invariant set of (1).

2 Mathematical Model

We have, N = S +I + Q, then,

$$\dot{N} = \rho + \lambda - \nu - \mu S - \mu_1 I - \mu_2 Q - dN$$

We replace S with $S = N - I - Q$; we obtain:

$$N = \rho + \lambda - \nu - (\mu + d)N - (\mu_1 - \mu)I - (\mu_2 - \mu)Q \tag{5}$$

Then the system (3) can be write as:

$$\begin{cases} \dot{I} = \dfrac{(\beta + k)(N - I - Q)\mathrm{I}}{1 + bI} - (\mu_1 + d + \gamma)I\,, \\ \dot{Q} = \gamma I - \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau) - (\mu_2 + d)Q\,, \\ \dot{N} = \rho + \lambda - \nu - (\mu + d)N - (\mu_1 - \mu)I - (\mu_2 - \mu)Q\,. \end{cases} \tag{6}$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection I=0, the system (6) has a disease-free equilibrium E_0.

$$\mathrm{E}_0 = (\hat{\mathrm{N}}, \hat{I}, \hat{Q})^T = \left(\frac{\rho + \lambda - \nu}{\mu + d}, 0, 0\right)^T. \tag{7}$$

Theorem 1

The disease-free equilibrium E_0 of the system (6) is locally asymptotically stable if $\mathrm{R}_0 < 1$ and E^*_τ is the unique positive endemic equilibrium point which exists if $\mathrm{R}_0 > 1$.

Proof

The eigenvalues can be determined by solving the characteristic equation of the linearization of (6) near E_0. Therefore, the eigenvalues are:

$$A_1 = -(\mu+d), A_2 = \frac{(\beta+k)(\rho+\lambda-\nu)}{\mu+d} - (\mu_1+d+\gamma), A_3 = -(\mu_2+d+\gamma e^{-\mu_2 \tau}). \quad (8)$$

In order to A_2 will be negative, and then we define the basic reproduction number of the infection R_0 as follows:

$$R_0 = \frac{\beta+k}{\mu_1+d+\gamma} \times \frac{\rho+\lambda-\nu}{\mu+d}. \quad (9)$$

If $R_0<1$, $A_2<0$.

We have $A_1<0$, $A_2<0$, and $A_3<0$, if $R_0<1$.

Then E_0 of the system (6) is locally asymptotically stable.

In the presence of infection $I \neq 0$, substituting in the system, Ω also contains a unique positive, endemic equilibrium

$E_\tau^* = \left(S_\tau^*, I_\tau^*, Q_\tau^*\right)^T, \forall i = 1,2,3.$ Where

$$\begin{cases} N_\tau^* = \frac{1}{\mu+d}\left[(\rho+\lambda-\nu) - a_2 I_\tau^*\right], \\ I_\tau^* = \frac{(R_0-1)(\mu_1+d+\gamma)}{(\mu_1+d+\gamma)b + (\beta+k)a_3}, \\ Q_\tau^* = a_1 I_\tau^*, \\ a_1 = \frac{\gamma}{\mu_2+d+\gamma e^{-\mu_2 \tau}}, \\ a_2 = \mu_1 - \mu + \frac{(\mu_2-\mu)\gamma}{\mu_2+d+\gamma e^{-\mu_2 \tau}}, \\ a_3 = \frac{a_2}{\mu+d} + 1 + a_1. \end{cases} \quad (10)$$

So E_τ^* is the unique positive endemic equilibrium point which exists if $R_0>1$.□

3 Mathematical Analysis

Lemma 1

The plane $N = \frac{\rho+\lambda-\nu}{\mu+d}$, is an invariant manifold of (6), which is attracting in the first octant.

Proof

We have, $N(t) = \frac{\rho+\lambda-\nu}{\mu+d}$, which is the solution of (6), then

$$\lim_{t\to\infty} N(t) = \frac{\rho+\lambda-\nu}{\mu+d}. \quad (11)$$

We reduce system (6), and then we have:

$$\begin{cases} \dot{I} = \dfrac{(\beta + k)(N - I - Q)\mathrm{I}}{1 + bI} - (\mu_1 + d + \gamma)I \triangleq L_1(I,Q), \\ \dot{Q} = \gamma I - \gamma \mathrm{e}^{-\mu_2 \tau}\mathrm{Q}(\mathrm{t} - \tau) - (\mu_2 + d)Q \triangleq L_2(I,Q), \end{cases} \tag{12}$$

Theorem 2

System (12) does not have nontrivial periodic orbits if $\dfrac{b(\mu_1 + d + \gamma)}{\beta + k} > -1$.

Proof

We have, system (12), for I>0 and Q >0. The dulac function [20], is the following,

$$D(I,Q) = \frac{1 + bI}{(\beta + k)\mathrm{I}} \tag{13}$$

Using (13) into the system (12) we obtain,

$$\frac{\partial(DL_1)}{\partial I} = -\left[1 + \frac{b(\mu_1 + d + \gamma)}{\beta + k}\right] \tag{14}$$

$$\frac{\partial(DL_2)}{\partial Q} = -\left[\frac{(1 + bI)(\mu_2 + d\gamma \mathrm{e}^{-\mu_2 \tau})}{(\beta + k)\mathrm{I}}\right] \tag{15}$$

In addition, (13) and (14) we obtain,

$$\frac{\partial(DL_1)}{\partial I} + \frac{\partial(DL_2)}{\partial Q} = -\left[1 + \frac{b(\mu_1 + d + \gamma)}{\beta + k}\right] - \left[\frac{(1 + bI)(\mu_2 + d\gamma \mathrm{e}^{-\mu_2 \tau})}{(\beta + k)\mathrm{I}}\right] \tag{16}$$

$$\frac{\partial(DL_1)}{\partial I} + \frac{\partial(DL_2)}{\partial Q} < 0. \tag{17}$$

If $\dfrac{b\left(\mu_1 + d + \gamma\right)}{\beta + k} > -1$

Theorem 3.

If $R_0 > 1$, then the endemic -disease point E_τ^* is globally asymptotically stable.

Proof

With the dulac function, in (13), and with the same proof into Theorem 2, we obtain (16).

Hence, according to (17), the system (12) has not periodic orbits. Since (12) admit only two equilibriums E_0 and E_τ^*. When R_0> 1 and E_0 is unstable, hence by Poincare- Binedixon theorem [20], E_τ^* is globally asymptotically stable.

3.1 Properties of Equilibriums

In order to study the properties of the disease-free equilibrium and the endemic equilibrium, we rescale (12), as following:

$$x = \frac{(\beta + k)\mathrm{I}}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}, y = \frac{(\beta + k)Q}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}, \tau = (\mu_2 + d + \gamma e^{-\mu_2 \tau})t. \tag{18}$$

Using (18) into system (12), we get a new system, which is define as follows:

$$\begin{cases} \frac{dx}{d\tau} = \frac{x}{1 + (\frac{\mu_2 + d + \gamma e^{-\mu_2 \tau}}{\beta + k})x} \times \left[\frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} - x - y \right] - (\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}})x, \\ \frac{dy}{d\tau} = (\frac{\gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}})x - y. \end{cases} \tag{19}$$

The system (19) has a disease-free equilibrium E_0, which is the same point of system (6).

The unique positive equilibrium (x^*, y^*) of system (19) is the endemic equilibrium E_τ^* of model (6) if and only if:

$$\frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} - (\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}) > 0 \tag{20}$$

Where,

$$\begin{cases} x^* = \frac{(\rho + \lambda - \nu)(\beta + k) - (\mu_1 + d + \gamma)}{(\mu_2 + d + \gamma e^{-\mu_2 \tau})(\frac{\mu_1 + d + \gamma}{\beta + k} + 1) + \gamma}, \\ y^* = \frac{\gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} x^*. \end{cases} \tag{21}$$

We first determine the stability and topological type of (0, 0). The Jacobian matrix of system (19) at (0, 0) is

$$M_0 = \begin{bmatrix} \frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} - (\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}) & 0 \\ \frac{\gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} & -1 \end{bmatrix}.$$

If $\frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} - (\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}) = 0$, then there exists a small neighborhood N_0 of (0, 0) such that the dynamics of system (19) are equivalent to that

$$\begin{cases} \frac{dx}{d\tau} = -x^2 + o((x, y)^2), \\ \frac{dy}{d\tau} = (\frac{\gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}})x - y. \end{cases} \tag{22}$$

By [20], we have (0, 0) is a saddle-node.

Theorem 4

The disease-free equilibrium (0, 0) of (19) is

1. a stable hyperbolic node if;
$$\left(\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}\right) - \frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} > 0,$$
2. a saddle-node if;
$$\left(\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}\right) - \frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} = 0,$$
3. a hyperbolic saddle if;
$$\left(\frac{\mu_1 + d + \gamma}{\mu_2 + d + \gamma e^{-\mu_2 \tau}}\right) - \frac{(\rho + \lambda - \nu)(\beta + k)}{\mu_2 + d + \gamma e^{-\mu_2 \tau}} < 0.$$

Theorem 5

Let R_0 be defined by (9)

1. If $R_0 < 1$, then system (6) has a unique disease-free equilibrium defined by (7), which is a global attractor in the first octant.
2. If $R_0 = 1$, system (6) has a unique disease-free equilibrium defined by (7), which attracts all orbits in the interior of the first octant.
3. If $R_0 > 1$, then model (6) has two equilibria, a disease-free equilibrium defined by (7) and an endemic equilibrium defined by (10), the latter who is a global attractor in the interior the first octant.

4 Stochastic Stabilities

The system (3) transformed to the Itô Stochastic differential equations. We replace (β+k) by (β+k) +ac (t) where c (t) is white noise.

$$\begin{cases} dS = \left[\rho + \lambda - \nu - (\mu + d)S - \frac{(\beta + k)SI}{1 + bI} + \gamma e^{-\mu_2 \tau} Q(t - \tau)\right] dt - a \frac{SI}{1 + bI} dc, \\ dI = \left[\frac{(\beta + k)SI}{1 + bI} - (\mu_1 + d + \gamma)I\right] dt + a \frac{SI}{1 + bI} dc, \\ dQ = \left[\gamma I - \gamma e^{-\mu_2 \tau} Q(t - \tau) - (\mu_2 + d)Q\right] dt. \end{cases} \quad (23)$$

Theorem 6

The set Ω is almost surely invariant by the stochastic system (23). Thus if $(S_0, I_0, Q_0) \in \Omega$,

then $P[(S, I, Q) \in \Omega] = 1$.

Proof

The system (23) implies that $dN \leq \left[\rho + \lambda - \nu - (\mu + d)N\right] dt$, and then we have for all $t \geq 0$;

$$N(t) \leq \left[\frac{\rho + \lambda - \nu}{\mu + d} + \left(N_0 - \frac{\rho + \lambda - \nu}{\mu + d}\right)\right] e^{-(\mu + d)t}$$

Since $(S_0, I_0, Q_0) \in \Omega$, and then we have for all $t \geq 0$;

$$N(\mathrm{t}) \leq \frac{\rho + \lambda - \nu}{\mu + d} \tag{24}$$

There exist $\varepsilon_0>0$, such that $S_0>\varepsilon_0>0$, $I_0>\varepsilon_0>0$ and $Q_0>\varepsilon_0>0$. Considering

$$\begin{aligned} T_\varepsilon &= inf\left\{t \geq 0,\ S(\mathrm{t}) \leq \varepsilon\ or\ I(\mathrm{t}) \leq \varepsilon\ or\ Q(\mathrm{t}) \leq \varepsilon,\right\},\ for\ \varepsilon \leq \varepsilon_0, \\ \mathrm{T} &= \lim_{t\to\infty} T_\varepsilon = inf\left\{t \geq 0,\ S(\mathrm{t}) \leq 0\ or\ I(\mathrm{t}) \leq 0\ or\ Q(\mathrm{t}) \leq 0\right\}, \end{aligned} \tag{25}$$

Let $V(\mathrm{t}) = log\left(\frac{\rho + \lambda - \nu}{(\mu + d)S(t)}\right) + log\left(\frac{\rho + \lambda - \nu}{(\mu + d)I(t)}\right) + log\left(\frac{\rho + \lambda - \nu}{(\mu + d)Q(t)}\right)$

Then, using Itô formula we have, for all $\mathrm{t} \geq 0$ and $U \in \left[t \wedge T_\varepsilon\right]$

$$dV(U) = \begin{bmatrix} -\frac{\rho + \lambda - \nu}{S(U)} + (\mu + d) - \gamma e^{-\mu_2 \tau}\frac{\mathrm{Q}(U)}{S(U)} + \frac{(\beta + k)\mathrm{I}(U)}{1 + bI(U)} \\ -\frac{(\beta + k)S(U)}{1 + bI} + (\mu_1 + d + \gamma) - \frac{\gamma I(U)}{Q(U)} + \gamma e^{-\mu_2 \tau} \\ +(\mu_2 + d) + \frac{1}{2}\frac{I^2(U)}{\left(1 + bI(U)\right)^2} + \frac{1}{2}\frac{S^2(U)}{\left(1 + bI(U)\right)^2} \end{bmatrix} dU + \left[a\frac{\mathrm{I}(U)}{1 + bI(U)} - a\frac{S(U)}{1 + bI(U)}\right] dc(U),$$

Then we have,

$$dV(U) \leq \begin{bmatrix} (\mu + d) + \frac{(\beta + k)\mathrm{I}(U)}{1 + bI(U)} + (\mu_1 + d + \gamma) + (\mu_2 + d) \\ + \gamma e^{-\mu_2 \tau} + \frac{1}{2}\frac{I^2(U)}{\left(1 + bI(U)\right)^2} + \frac{1}{2}\frac{S^2(U)}{\left(1 + bI(U)\right)^2} \end{bmatrix} dU + \left[\frac{a\left(\mathrm{I}(U) - S(U)\right)}{1 + bI(U)}\right] dc(U) \tag{26}$$

With (24), we have $S(\mathrm{U}), I(\mathrm{U}), Q(\mathrm{U}) \in \left[0, \frac{\rho + \lambda - \nu}{\left(\mu + d\right)}\right]$.

For all $U \in \left[t \wedge T_\varepsilon\right]$,

$$M_1 = \mu + \mu_1 + \mu_2 + 3d + \gamma + \gamma e^{-\mu_2 \tau} + \frac{(\beta + k)(\rho + \lambda - \nu)}{\mu + d} + \left(\frac{\rho + \lambda - \nu}{\mu + d}\right)^2, \tag{27}$$

We replace (27) into (26), then

$$dV(U) \leq M_1 dU + \left[\frac{a\left(\mathrm{I}(U) - S(U)\right)}{1 + bI(U)}\right] dc(U) \tag{28}$$

With integration to (28), we obtain

$$V(\mathrm{U}) \leq M_1 U + a\int_0^U \left[\frac{\left(\mathrm{I}(w) - S(w)\right)}{1 + bI(w)}\right] dc(w)$$

By [17], we have $\int_0^U \left[\frac{\left(\mathrm{I}(w) - S(w)\right)}{1 + bI(w)}\right] dc(w)$ is a mean zero process, then,

$$E(V(U)) \leq M_1 U \tag{29}$$

For all t ≥ 0 and $U \in \left[t \wedge T_{\varepsilon}\right]$,

$$S(t \wedge T_{\varepsilon}), I(t \wedge T_{\varepsilon}), \ and \ \ Q(t \wedge T_{\varepsilon}) \in [0, \frac{\rho + \lambda - \nu}{\mu + d}]$$

Then

$$0 \leq E\left(V(t \wedge T_{\varepsilon})\right) \leq M_1(t \wedge T_{\varepsilon}) \leq M_1 t$$

Where $\Psi_{[T_{\varepsilon} \leq t]}$ is the indicator function of a $\left[T_{\varepsilon} \leq t\right]$.

$$\begin{aligned} &E\left(V(t \wedge T_{\varepsilon})\right) \geq E\left(V(t) \times \Psi_{[T_{\varepsilon} \leq t]}\right) + E\left(V(t) \times \Psi_{[T_{\varepsilon} > t]}\right) \\ &E\left(V(t \wedge T_{\varepsilon})\right) \geq E\left(V(t) \times \Psi_{[T_{\varepsilon} \leq t]}\right) \geq P\left(T_{\varepsilon} \leq t\right) \log\left(\frac{\rho + \lambda - \nu}{(\mu + d)\varepsilon}\right) \end{aligned} \tag{30}$$

Combine (29) and (30), we obtain for all t ≥ 0, the following

$$P(T_{\varepsilon} \leq t) \leq \frac{M_1 t}{\log\left(\frac{\rho + \lambda - \nu}{(\mu + d)\varepsilon}\right)} \tag{31}$$

For all $t \geq 0, \varepsilon \to 0$, $P(T \leq t) = 0$,

$P(T \leq \infty) = 0$.□

Theorem 7

If $(\beta + k)^2 - 2a^2(\mu + d) < 0$, S (t) converge exponentially almost surely to $\frac{\rho + \lambda - \nu}{\mu + d}$.

Proof

We use Itô formula to the first equation in system (23), we obtain

$$d \log\left|S - \frac{\rho + \lambda - \nu}{\mu + d}\right| = \left[\begin{array}{l} \frac{\rho + \lambda - \nu}{S - \frac{\rho + \lambda - \nu}{\mu + d}} - \frac{(\mu + d)S}{S - \frac{\rho + \lambda - \nu}{\mu + d}} - \frac{(\beta + k)}{1 + bI} \times \frac{SI}{S - \frac{\rho + \lambda - \nu}{\mu + d}} + \frac{\gamma e^{-\mu_2 \tau} Q}{S - \frac{\rho + \lambda - \nu}{\mu + d}} \\ - \frac{1}{2} \times \left(\frac{aSI}{(1 + bI)\left(S - \frac{\rho + \lambda - \nu}{\mu + d}\right)}\right)^2 \end{array}\right] dt$$

$$- \left[\frac{a}{S - \frac{\rho + \lambda - \nu}{\mu + d}} \times \frac{SI}{1 + bI}\right] dc,$$

$$d \log\left|S - \frac{\rho + \lambda - \nu}{\mu + d}\right| = \left[\begin{array}{l} -(\mu + d) - \frac{(\beta + k)}{1 + bI} \times \frac{SI}{S - \frac{\rho + \lambda - \nu}{\mu + d}} + \frac{\gamma e^{-\mu_2 \tau} Q}{S - \frac{\rho + \lambda - \nu}{\mu + d}} \\ - \frac{1}{2} \times \left(\frac{aSI}{(1 + bI)\left(S - \frac{\rho + \lambda - \nu}{\mu + d}\right)}\right)^2 \end{array}\right] dt$$

$$- \left[\frac{a}{S - \frac{\rho + \lambda - \nu}{\mu + d}} \times \frac{SI}{1 + bI}\right] dc,$$

Then

$$d\log\left|S-\frac{\rho+\lambda-\nu}{\mu+d}\right|=\left[\begin{array}{l}-(\mu+d)-(\beta+k)\times\dfrac{SI}{(1+bI)\left(S-\dfrac{\rho+\lambda-\nu}{\mu+d}\right)}\\ -\dfrac{1}{2}a^2\times\left(\dfrac{SI}{(1+bI)\left(S-\dfrac{\rho+\lambda-\nu}{\mu+d}\right)}\right)^2\end{array}\right]dt$$
$$-\left[\frac{a}{S-\dfrac{\rho+\lambda-\nu}{\mu+d}}\times\frac{SI}{1+bI}\right]dc,$$

We suppose that

$$G(y)=-\frac{1}{2}a^2y^2-(\beta+k)y-(\mu+d),y=\frac{SI}{(1+bI)\left(S-\dfrac{\rho+\lambda-\nu}{\mu+d}\right)}. \tag{32}$$

If the determinant of the equation is negative, then for all x.

$$G(y)\le\frac{\Delta}{a^2}dt,\text{with}\Delta=(\beta+k)^2-2a^2(\mu+d) \tag{33}$$

We have

$$d\log\left(S-\frac{\rho+\lambda-\nu}{\mu+d}\right)\le\frac{\Delta}{a^2}dt-\left[\frac{a}{S-\dfrac{\rho+\lambda-\nu}{\mu+d}}\times\frac{SI}{1+bI}\times\right]dc.$$

With integration, we obtain

$$\log\left(S-\frac{\rho+\lambda-\nu}{\mu+d}\right)\le\frac{\Delta}{a^2}dt-a\int_0^t\frac{S(w)}{1+bI(w)}\times\frac{I(w)}{S(w)-\dfrac{\rho+\lambda-\nu}{\mu+d}}dc(w).$$

Since

$$\lim_{t\to\infty}\int_0^t\frac{S(w)}{1+bI(w)}\times\frac{I(w)}{S(w)-\dfrac{\rho+\lambda-\nu}{\mu+d}}dc(w)=0\text{ , almost surely.}$$

Therefore

$$\lim_{t\to\infty}\sup\frac{1}{t}\log\left(S-\frac{\rho+\lambda-\nu}{\mu+d}\right)\le\frac{\Delta}{a^2} \tag{34}$$

S (t) is exponentially almost stable.□

5 Conclusion

This paper addresses an epidemic nonlinear model with temporary immunity and saturated incidence rate, whenever the quarantine individuals will return to the susceptible. We study mathematical model with

system (6) which have a disease free equilibrium E_0 defined in (7) and the endemic equilibrium E_τ^* defined in (10). It founded that the disease free equilibrium to system (6) is locally asymptotically stable if, $R_0 < 1$, and the existence of endemic equilibrium if $R_0 > 1$, with the basic reproduction number of the infection R_0 is defined in (9). The analysis mathematical study the reduce model in (12), which does not have non trivial periodic orbits in theorem 2, under condition, and theorem 3 who says that the endemic -disease point E_τ^* is globally asymptotically stable, if $R_0 > 1$, and study the properties of equilibriums to the model (19) with theorems under some conditions. Finally stochastic stability of system (23), which study theorem 6 who says that Ω is almost surely invariant thus if (S_0, I_0, Q_0) ∈ Ω, then P [(S, I, Q) ∈Ω] = 1, and theorem 7 which proof that , S (t) converge exponentially almost surely to $\frac{\rho + \lambda - \nu}{\mu + d}$ under condition.

Competing Interests

Author has declared that no competing interests exist.

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