**Complemented semiring satisfying the identity u + 1 = 1 + u = u**

**ABSTRACT:** This paper contains some results on complemented semirings involving one variable. We consider complemented semiring (S1, +, .) satisfying the identity u + 1=1 + u = u for all u, v in S1 then it is proved that (S1, +) is quasi, weakly, separative and also (S1, +, .) is an idempotent. For this complemented semiring we also determine the additive and multiplicative structures. Complemented semiring is holds and equivalent many results by using the identity u + 1 = 1 + u = u for all u, v in S1.

**KEYWORDS:** complemented semiring, quasi-separative, weakly separative, rectangular band, idempotent.

**AMS Subject Classification(2010): 16Y60**

**INTRODUCTION:** Algebraic systems are artistic with a partially or fully ordered met within several disciplines of Mathematics. In recent years interest in the study of partially ordered and fully ordered semigroups, groups, semirings, semi modules, rings and fields has been increasing enormously. The theory of semigroups had essentially two origins. One was an attempt to generalize both group theory and ring theory to the algebraic system consisting of a single associated operation which from the group theoretical point of view omits the axioms of the existence of the identities and inverses and from the ring theoretical point of view omits the additive structure of the ring.

However in semirings it is possible to derive the additive and multiplicative structures from their special multiplicative structures and vice-versa. The semiring identities are taken from the book of Jonathan S. Golan[4], entitiled “Theory of semirings with applications in mathematics and theoretical computer science”. In this paper we investigate the additive and multiplicative properties of complemented semirings.

**Theorem 1.1:** Let (Sl , +, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl, if (Sl , .) be a rectangular band then (Sl , +) is

1. Quasi – separative.
2. Weakly – separative.
3. Separative.

Proof:Let (Sl , +, .) be a complemented semiring in which (Sl , .) be a rectangular band.

Assume that Sl satisfies the identity u + 1 = 1 + u = u for any u Є Sl.

Now, for all u, v Є Sl, such that

1. Consider, u + u = u + v

 u.1 + u = 1.u + v

 u(u + v) + u = (u + v)u + v

 u(u + v + 1) = u.u + v.u + v

 u(1 + 1) = u + v.u + v

 u.1 = u(1+ v) + v

 u = uv + v

 u = 0 + v

 u = v

Implies, (Sl , +) is quasi-separative.

1. Let u + v = u + v.1

 u + v = u + v(u + v)

 u + v = u + vu + vv

 u + v = (1 + v)u + vv

 u + v = vu + vv

 u + v = v(1 + u) + vv

 u + v = v + vu + vv

 u + v = v + (1 + v)u + v

 u + v = v + u + vu + v

 u + v = v + u + v(u + 1)

 u + v = v + u + vu

 u + v = v + u + 0

 u + v = v + u (1.1)

From (i) and (ii) u + u = u + v = v + u = v + v. (1.2)

Implies, u = v.

Implies, (Sl , +) is weakly separative.

1. From (1.2), we have, u + u = u + v = v + v implies u = v (1.3)

From (1.1) we have, u + v = v + u implies u = v (1.4)

From (1.3) and (1.4)

(Sl , +) is separative semigroup.

**Theorem 1.2.** Let (Sl , +, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl, then for any v Є Sl the following conditions holds,

1. u + v + 1 = uv
2. uv + 1 = uv

**Proof.** Let (Sl , +, .) be a complemented semiring for all u, v Є Sl such that u + v = 1 and uv = vu = 0 and (Sl , +) satisfying the identity u + 1 = 1 + u = u for all u Є Sl .

For all u, v Є Sl.

1. Consider, u + v + 1 = u + 1.v + 1

 u + v + 1 = u + (u + v)v + 1

 u + v + 1 = u + uv + vv + 1

 u + v + 1 = u + uv + vv + u + v

 u + v + 1 = u + uv + vv + v + u

 u + v + 1 = u + uv + (v + 1)v + u

 u + v + 1 = u + uv + vv + u

 u + v + 1 = u + uv + (v + u)

 u + v + 1 = u + uv + 1

 u + v + 1 = (u + 1) + uv

 u + v + 1 = u + uv

 u + v + 1 = u(1 + v)

 u + v + 1 = u v

1. Let consider, uv + 1 = uv + u + v

 uv + 1 = u(v + 1) + v

 uv + 1 = uv + v

 uv + 1 = (u + 1)v

 uv + 1 = uv

**Theorem 1.3.** Let (Sl , +, .) be a complemented semiring then for any u, v Є Sl then the following statements are equivalent.

1. u + 1 = u.
2. un + 1 = un.
3. (uv)n + 1 = (uv)n for all u, v Є Sl.

**Proof.**Let (Sl , +, .) be a complemented semiring then for any u, v Є Sl.

Let u + 1 = 1 + u = u, for all u Є Sl then we have to prove that un + 1 = unfor some positive integers n.

We prove by mathematical induction method.

Case(i):

 If n = 1 then u1 + 1 = u

 u + 1 = u

Implies, given statement is true for n = 1

Case(ii):

 Assume that given result is true for n = k, i.e., uk + 1 = uk , for some positive integer k.

To prove that the result is true for n = k + 1.

 uk + 1 + 1 = uk .u + 1

 uk + 1 + 1 = uk .u + (u + v)

 uk + 1 + 1 = (uk + 1)u + v.1

 uk + 1 + 1 = (uk + 1)u + v.(u + v)

 uk + 1 + 1 = uk u + v.u + v.v

 uk + 1 + 1 = uk u + v.u + v

 uk + 1 + 1 = uk u + v.(u + 1)

 uk + 1 + 1 = uk u + v.u

 uk + 1 + 1 = uk u + 0

 uk + 1 + 1 = uk u

 uk + 1 + 1 = uk + 1

Hence the result is true for n = k + 1.

Therefore, u1 + 1 = u

 uk + 1 = uk

If (ii) implies (iii)

Let un + 1 = un and vn + 1 = vn for all u, v Є Sl.

Consider, (uv)n + 1 = un vn + 1

 (uv)n + 1 = un vn + u + v

 (uv)n + 1 = un vn + u.1 + v

 (uv)n + 1 = un vn + u(u + v) + v

 (uv)n + 1 = un vn + u.u + u.v + v

 (uv)n + 1 = un vn + u.u + (u + 1)v

 (uv)n + 1 = un vn + u.u + u.v

 (uv)n + 1 = un vn + u + u.v

 (uv)n + 1 = un vn + u(1 + v)

 (uv)n + 1 = un vn + u.v

 (uv)n + 1 = un vn + 0

 (uv)n + 1 = un vn

 (uv)n + 1 = (uv)n

If (ii) implies (i)

Let un + 1 = un for any u Є S1 and some positive integer n, if n = 1, then

u1 + 1 = u => u + 1 = u => un + 1 = un => u + 1 = u.

**Theorem 1.4.** Let (Sl , +, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl, then for any v Є Sl if u + c = v and v + d = u then u = v.

**Proof.** Let (Sl , +, .) be a complemented semiring for all u, v Є Sl.

Consider, u = u.1

 u = u.(u + v)

 u = u.(v + d + u + c)

 u = u.v + u.d +u.u + u.c

 u = u.v + u.u + u.d + u.c

 u = u.v + u + u.d + u.c

 u = u(v + 1) + u.d + u.c

 u = u.v + u.d + u.c

 u = u(v + d) + u.c

 u = u(v + d + c)

 u = u(u + c)

 u = uv

 u = (u + 1)v

 u = u.v + v

 u = u.v + v

 u = 0 + v

 u = v

**Theorem 1.5.** Let (Sl ,+, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl and for any v Є S then (Sl , +, .) is an idempotent.

**Proof.** Let (Sl ,+, .) be a complemented semiring for all u, v Є Sl such that u + v = 1 and uv = vu =0 (Sl , +) satisfying the identity u + 1 = 1 + u = u for all u Є Sl .

For all u, v Є Sl.

Consider,

 u + v = 1

 pre multiply by u on both sides

 u(u +v) = u.1

 u.u + u.v = u

 u(1 + u) + uv = u

 u + u.u + uv = u

 u + u(u + v) = u

 u + u(1) = u

 u + u = u

Thus (Sl , +) is an idempotent.

Again consider,

 u + v = 1

 pre multiply by u on both sides

 u(u + v) = u.1

 u.u + u.v = u

 u.u + u(1+v) = u

 u.u + u1+ uv = u

 u.u + u + uv = u

 u.u + u + 0 = u

 u.(u + 1) = u

 u.u = u

Thus (Sl , .) is an idempotent.

Therefore (Sl , +, .) is an idempotent.

**Theorem 1.6.** Let (Sl ,+, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl and for any v Є Sl  then

1. u + v2 = v
2. v2 + u = v

**Proof.** Let (Sl ,+, .) be a complemented semiring for all u, v Є Sl such that u + v = 1 and uv = vu =0 and (Sl , +) satisfying the identity u + 1 = 1 + u = u for all u Є Sl .

For all u, v Є Sl.

1. Consider, u + v2 = u + v.v

 = u + v(1 + v)

 = u + v + v2

 = 1 + v2

 = 1 + v.v

 = 1 + v(1 + v)

 = 1 + v + v2

 = v + v2

 = v + v.v

 = v + v

 = v

1. Consider v2 + u = v.v + u

 = (v + 1)v + u

 = v2 + v + u

 = v2 + u + v

 = v2 + 1

 = v.v + 1

 = v.(1 + v) + 1

 = v + v2 + 1

 = v + 1 + v2

 = v + v2

 = v + v.v

 = v + v

 = v

**Theorem 1.7.** Let (Sl , +, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl  and for any v Є Sl then u + v + u = 0 ⇒ u = 0 = v for all u, v Є Sl.

**Proof.** Let (Sl , +, .) be a complemented semiring for all u, v Є Sl such that u + v = 1 and uv = vu =0 and (Sl , +) satisfying the identity u + 1 = 1 + u = u for all u Є Sl .

For all u, v Є Sl .

Consider, u + v + u = 0 (4.2.5)

 1 + u = 0

 u = 0

Since u = 1 to equation (4.2.5)

Then, 1 + v + 1 = 0

 v + 1 = 0

 v = 0

Thus u = 0 = v for all u, v Є S1.

**Theorem 1.8.** Let (Sl , +, .) be a complemented semiring satisfying the identity u + 1 = 1 + u = u for all u Є Sl and for any v Є Sl, then uvu = 1 and vuv = 1 implies u = 1 = v for all u, v Є Sl.

**Proof.** Let (Sl , +, .) be a complemented semiring for all u, v Є Sl such that u + v = 1 and uv = vu =0 and (Sl , +) satisfying the identity u + 1 = 1 + u = u for all u Є Sl.

 Let any u, v Є Sl then,

Consider, uvu = 1

 (u + 1)vu = 1

 uvu + vu = 1

 v + vu = 1

 v(1 + u) = 1

 vu = 1

 (1 + v)u = 1

 u + vu = 1

 u + 0 = 1

 u = 1

Again consider, uvu = 1

 uv(u + 1) = 1

 uvu + uv = 1

 v + uv = 1

 v + 0 = 1

 v = 1

Thus u = 1 = v for all u, v Є Sl.

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