Modern Digital Control

This chapter focuses on the theme of modern digital control for modern digital control is a main trend and control fashion in present industry. In this chapter， the basic conceptions on modern digital control are narrated and explained to understand modern digital control system. Then stability and steady-state error are analyzed and deduced. Beside these, transient response, gain and nonlinear state observer are also introduced to further analyze a given digital control system. Furthermore, some key elements and applications in digital control are narrated and enumerated.

Objectives

When you have finished this chapter, you should be able to:

- Know popular reason of modern digital control and its some basic concepts.
- Draw the block diagram of digital control of a given analogue system
- Learn how to analyze the stability and observability of digital control system.
- Design or select a digital compensator.
- Recognize some main applications of digital control.

11.1 Introduction

The use of a digital computer as a controller device has grown quickly during the past several decades as the price and reliability of digital computers have been improved dramatically. The structure diagram of digital control system is shown in figure11.1.

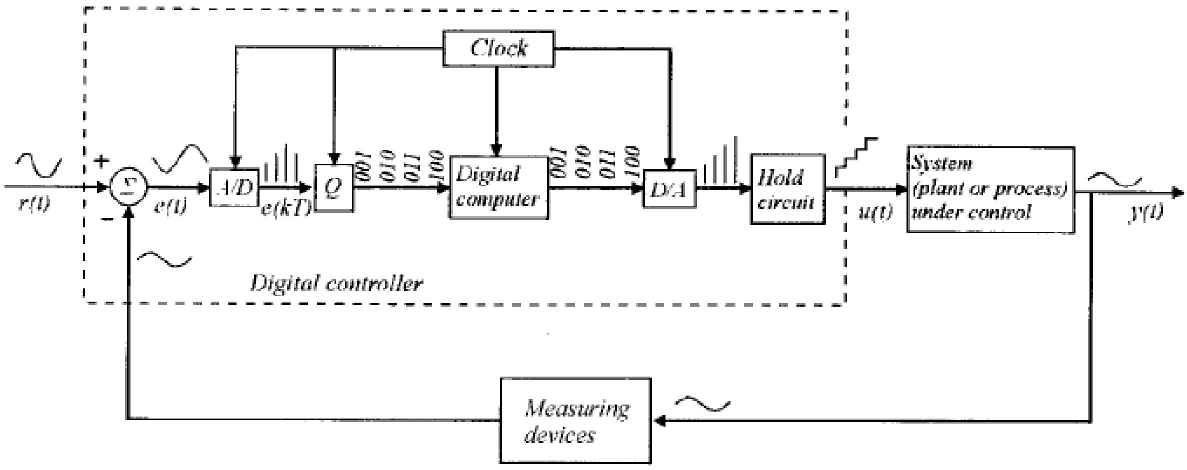


Figure11.1 Block diagram of a typical closed-loop digital control or computer control system

The system under control is a continuous-time system(i.e., motor, robot, electrical power plant etc). The “heart” of the controller is a digital computer. This digital computer receives and operates on signals in digital form, as contrasted to continuous signals. In nature, a digital control system uses digital signals and a digital computer to control a process. The measurement data is converted from analogue signal to digital signal. This course is a A/D converting course. A A/D converter converts a continuous-time signal into a discrete-time signal at times specified by a clock. After processing the input, digital controller provides an output in digital form. This output is then converted into analog signal by D/A converter. The D/A converter, in contrast, converts the discrete-time signal output of the computer to a continuous-time signal to be fed to the system. The D/A converter normally includes a sampling hold circuit. The quantizer Q converts a discrete-time signal to binary digits.

The controller may be designed to satisfy simple, as well as complex, specifications. Hence, it may operate as a simple logic device as in programmable logic controllers(PLCs), or make dynamic and complicated processing operations on the error between desired input and actual output to produce a suitable input to control the system. This control input $u(t)$ to the actuating system must be such that the behavior of the closed-loop system satisfies desired specifications.

The digital control systems are used in many applications: machine tools, metal processes, chemical processes, aircraft control and automobile traffic control,automobile, radar tracking systemand so on. At present, the digital control systems are changing our life and manufacturing mode in industrial fields.

The problem of realizing a digital controller is mainly one of developing a computer program. Digital controllers present significant advantages over classical analog controllers. Some of these advantages are as follows:

(1) Digital controllers have greater flexibility in modifying the controller's content. Indeed, the controller's content may be confirmed by programming. For classical analog controller, its content has been fixed, and its content is not almost adjusted. If do, there will be expensive cost and complex procedures.

(2) Data processing is very simple. Complex computations may be performed in a fast and convenient and uniform way. Analog controller do not have such convenience.

(3) Digital controllers are superior over analog controllers with regard to the following characteristics:

A. Sensitivity

B. Drift effects

C. Internal noise

D. Reliability.

E. Digital sensor and actuator

F. Microprocessor

G. Flexible algorithm configuration

(4) Digital controller is cheaper than analog controller. Though the price of plenty of goods and service is increasing, the cost of digital circuit continues to decrease. Advance in very large scale integration technology makes it possible to fabricate better, cheaper, and

more reliable integration chip including digital controller and to offer them to consumer at a lower price. This makes the use of digital controller become more economical and more convenient and more reliable.

(5) Digital controllers are considerably smaller in size than analog controller.

(6) Digital controllers are more accurate than analog controllers. As we know, digital signals are represented in terms of zeros and ones with typically 12 bits or more to represent a single number. This involves a very small error as compared to analog signals where noise and power supply drift are always present. Furthermore, the digital processing of control signals involves addition and multiplication by stored numerical values. The errors that result from digital representation and arithmetic are negligible. By contrast, the processing of analog signals is performed using components such resistors and capacitors with actual values that vary significantly from the nominal design vales.

However, digital controllers have certain disadvantages compared with analog controllers. The most significant disadvantages is due to the error introduced during sampling analog signals, as well as during the quantization of discrete-time signals.

The mathematical background necessary for the study of digital control systems is the Z-transform which has been narrated in chapter2. The Z-transform facilitates the analysis and design of discrete-time control system in an analogous way to that Laplace transform does for the continuous-time control systems. Hence, before further learning the content of chapter11, suggest that readers should review the Z-transform given in chapter2.

11.2 Description and Analysis of Digital Control Systems

11.2.1 Description of linear time-invariant discrete-time systems

Linear discrete-time system means that system could operate directly with discrete-time signals. In this case, the input as well as output of system is discrete-time signal. A well-known discrete-time system is digital computer. In this case the input $u(k)$ and output $y(k)$ are number sequence (usually 0 and 1). Such type of system will be described in difference equation. The block diagram of discrete-time system is shown in figure11.2.

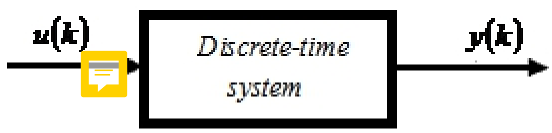


Figure11.2 The block diagram of discrete-time system

Besides input and output in figure11.2, there are data-converting parts such as A/D and D/A parts in digital controller, and the analogue-to-digital part in measurement device. These data-converting parts are called data-sampling system which is used to implement the conversion between analogue and digital signals.

From the viewpoint of mathematical expression, the discrete-time system description implies the determination of a law which assigns a output sequence $y(k)$ to a given input sequence $u(k)$ seen in figure11.3. The specific law connecting the input sequence $u(k)$ and output sequence $y(k)$ constitutes mathematical model of a discrete-time system. Symbolically, this relation can be expressed as follows:

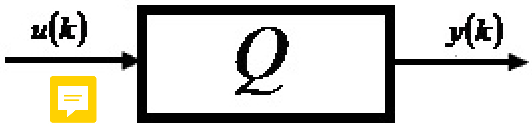


Figure11.3 Block diagram of discrete-time system

$$y(k) = Q(u(k)) \tag{11.1}$$

Here, sign Q is a discrete operator. Discrete-time system has plenty of characteristics, here some interesting and special characters are listed out below:

1. Linearity

A discrete-time system is linear if the following relation is found:

$$Q[c_1u_1(k)+c_2u_2(k)] = c_1Q[u_1(k)]+c_2Q[u_2(k)] = c_1y_1(k)+c_2y_2(k) \tag{11.2}$$

For every c_1, c_2 and $u_1(k)$, $u_2(k)$, where c_1, c_2 are constants and $y_1(k) = Q[u_1(k)]$ is the output of the system with input $u_1(k)$ and $y_2(k) = Q[u_2(k)]$ is the output of the system with input $u_2(k)$.

2.Time-invariant system

A discrete-time system is time-invariant if the following relation for every k_0 holds true:

$$Q[u(k-k_0)] = y(k-k_0) \tag{11.3}$$

Equation(11.3) shows that when the input to the system is shifted by k_0 units, the output of the system is also shifted by k_0 units.

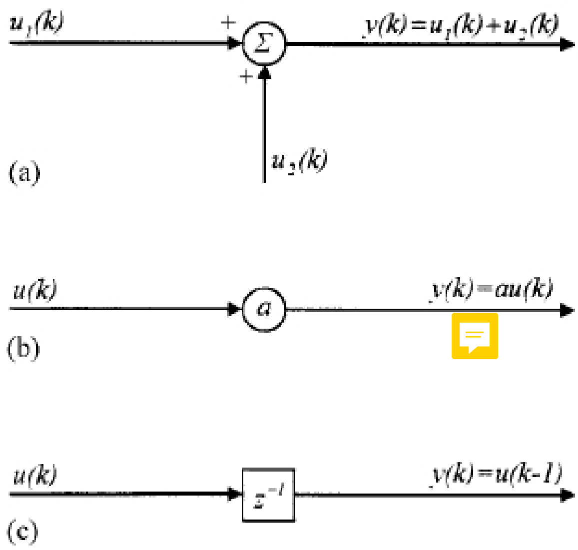


Figure11.4 (a) summation units, (b) amplification units, (c) delay units.

3.Causality

When the input of discrete system $u(k)=0$ for $k<k_0$, a discrete-time system is called causal if the output $y(k)=0$ for $k<k_0$. If a causal input $u(k)$ causes a causal response(output) $y(k)$, such discrete-time system is causal.

A linear time-invariant causal discrete-time system involves the following components: summation units, amplification units, and delay units. The block diagram of all three basic components is shown in figure11.4. The delay unit is the designated as z^{-1}, meaning that the output of system is identical to the input delayed by a time unit.

When these three components are suitably combined and interconnected, a discrete-time system may be formed, for example, the first-order discrete-time system shown in figure11.5. According to the summation of three signals, we can write out the following relation:

$$y(k)=b_0u(k)+b_1u(k-1)-a_1y(k-1)$$

Namely,

$$y(k)+a_1y(k-1)=b_0u(k)+b_1u(k-1) \tag{11.4}$$

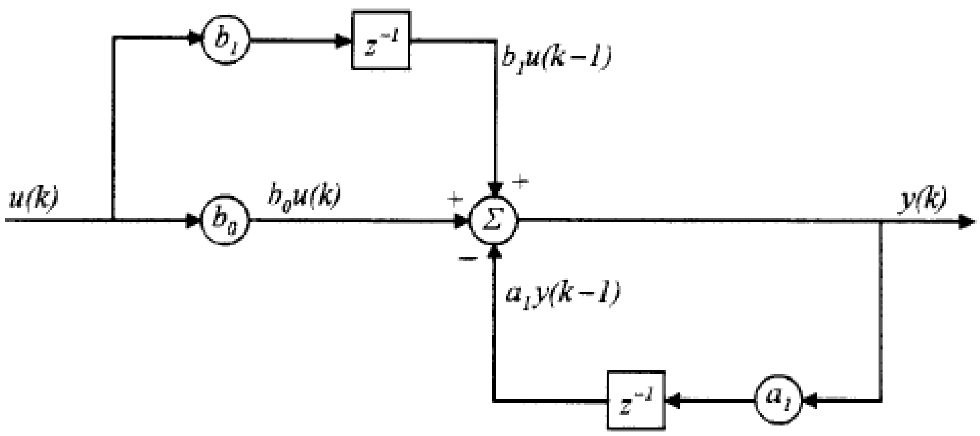


Figure11.5 Block diagram of first-order discrete-time system

As we have known in chapter2, the difference equation may be utilized to describe discrete-time system, which is only one kind of ways describing discrete system such as transfer function, state space model, impulse response and so on. To some extent, discrete-time system is only a special case of continuous-time system.

These four describing methods will be selected and reviewed for discrete-time system since they have been given in chapter2.

1. Difference equation

This describing method is very common. The general form of a difference equation is as follows:

$$y(k)+a_1y(k-1)+\cdots+a_ny(k-n)=b_0u(k)+b_1u(k-1)+\cdots+b_mu(k-m) \tag{11.5}$$

The initial conditions are $y(-1)$, $y(-2)$, $\cdots$, $y(-n)$. The solution of equation (11.5) may be found either in the time domain(employing method is similar to those for solving a differential equation in the time domain) or in the complex frequency or Z-domain using the Z-transform.

2. Transfer function

The transfer function of a discrete-time system is denoted by $H(z)$ and is defined as the ratio of the Z-transform functions of output $y(k)$ and input $u(k)$ under the condition $y(k)=u(k)=0$ for all negative values of k namely $k<0$. That's

$$H(z)=\frac{Z[y(k)]}{Z[u(k)]}=\frac{Y(z)}{U(z)}, \text{ where } y(k)=u(k)=0 \text{ for } k<0 \tag{11.6}$$

The transfer function of system described by the difference equation(11.5), with $y(k)=u(k)=0$ for $k<0$, is determined as follows:

The both sides of equation(11.5) are multiplied by the term z^{-k} and same terms are added and merged in k difference equations for $k=0,1,2,\cdots,\infty$ to yield

$$\begin{aligned}&\sum_{k=0}^{\infty} y(k)z^{-k}+a_1\sum_{k=0}^{\infty} y(k-1)z^{-k}+a_2\sum_{k=0}^{\infty} y(k-2)z^{-k}+\cdots+a_n\sum_{k=0}^{\infty} y(k-n)z^{-k}\\&=b_0\sum_{k=0}^{\infty} u(k)z^{-k}+b_1\sum_{k=0}^{\infty} u(k-1)z^{-k}+\cdots+b_m\sum_{k=0}^{\infty} u(k-m)z^{-k}\end{aligned} \tag{11.7}$$

Using the Z-transform time-shifting property and the assumption $y(k)=u(k)=0$ for $k<0$, above equation(11.7) can be simplified by Z-transform definition as follows:

$$Y(z)+a_1z^{-1}Y(z)+a_2z^{-2}Y(z)+\cdots+a_nz^{-n}Y(z)=b_0U(z)+b_1z^{-1}U(z)+\cdots+b_mz^{-m}U(z) \tag{11.8}$$

Hence, in terms of definition in equation(11.6), equation(11.8) can be written into the following rational polynomial form:

$$\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_mz^{-m}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}} \tag{11.9}$$

3. Impulse response

The impulse response of a system is denoted by $h(k)$ and is defined as the output of a system when its input is the unit impulse sequence $\delta(k)$ under the constraint that the initial conditions $y(-1)$, $y(-2)$, $\cdots$, $y(-n)$ are zero. The block diagram definition of the impulse response is shown in figure11.6.

Figure11.6 Block diagram of discrete-time impulse response

The transfer function $H(z)$ and the weight function $h(k)$ are related by the following equation:

$$H(z)=Z[h(k)] \tag{11.10}$$

Equation(11.10) indicates that $H(z)$ is the Z-transform of function $h(k)$ (it is called weight function, too).

4. State-space equation

State-space equation or simple state equations are a set of first-order difference equations which are used to describe high-order systems and which have the following form:

$$x(k+1) = Ax(k) + Bu(k) \tag{11.11a}$$

$$y(k) = Cx(k) + Du(k) \tag{11.11b}$$

$u(k) \in R^m$, $x(k) \in R^n$ and $y(k) \in R^p$, are the input, state, and output vectors respectively. And A, B, C and D are constant matrices with different dimensions.

Here, we postulate $Y(z) = Z[y(k)]$ and $U(z) = Z[u(k)]$. Then , the transfer function matrix $H(z)$ in equation(11.11) may be gained by

$$H(z) = C(zE_{unit} - A)^{-1} B + D \tag{11.12}$$

The impulse response matrix $h(k)$ is given by

$$h(k) = Z^{-1}[H(z)] = \begin{cases} D, & for \quad k = 0 \\ CA^{k-1}B, & for \quad k > 0 \end{cases} \tag{11.13}$$

11.2.2 Analysis of linear time-invariant discrete-time systems

On the basis of different description methods above, the analysis of linear time-invariant discrete-time system will be done, which each analyzing method corresponds to one of four description models.

1. Analysis of difference equation

Here, we will utilized an example to illustrate the analysis of difference equation.

Example11.1 A discrete -time system is described by the following difference equation with the initial condition $y(-1)$.

$$y(k) = bu(k) + ay(k-1)$$

Please solve the difference equation.

Solution

The difference equation may be solved to determine $y(k)$ by using the Z-transform as follows. Take the Z-transform of both sides of the equation to yield

$$Z[y(k)] = bZ[u(k)] + aZ[y(k-1)]$$

Or

$$Y(z) = bU(z) + a[z^{-1}Y(z) + y(-1)]$$

Thus

$$Y(z) = \frac{bU(z) + ay(-1)}{1 - az^{-1}} = \frac{z[bU(z) + ay(-1)]}{z - a} = \frac{bU(z)z}{z - a} + \frac{ay(-1)z}{z - a}$$

Assume that the excitation $u(k)$ is the unit step sequence, then the Z-transform of excitation $u(k)$ is the following:

$$U(z) = Z[u(k)] = \frac{z}{z-1}$$

Hence,

$$Y(z) = \frac{bU(z)z}{z-a} + \frac{ay(-1)z}{z-a} = \frac{bz^2}{(z-1)(z-a)} + \frac{ay(-1)z}{z-a}$$
$$= \frac{ay(-1)z}{z-a} + \left[\frac{b}{1-a}\right]\left[\frac{z}{z-1}\right] - \left[\frac{ab}{1-a}\right]\left[\frac{z}{z-a}\right]$$

Take the inverse Z-transform to yield

$$y(k) = a^{k+1}y(-1) + \frac{b}{1-a} - \frac{b}{1-a}a^{k+1}$$

When $|a|<1$, the output $y(k)$ clearly trends to convergence. The initial condition $y(k)$ only affects the transient response.Hence, the steady-state output y_{ss} can be obtained as follows:

$$y_{ss}(k) = \frac{b}{1-a}$$

2. Analysis of transfer function

The input $U(z)$, output $Y(z)$, and transfer function $H(z)$ are related by the following equation:

$$Y(z) = H(z)U(z)$$

Hence,

$$y(k) = Z^{-1}[Y(z)] = Z^{-1}[H(z)U(z)] \tag{11.14}$$

Steady-state output:

$$y_{ss}(k) = \lim_{k\to\infty} y(k) = \lim_{k\to\infty} Z^{-1}[H(z)U(z)] \tag{11.15}$$

3. Analysis of impulse response

The input $u(k)$, output $y(k)$, and transfer function $h(k)$ are related by the following convolution equation:

$$y(k) = u(k) * h(k) = \sum_{i=-\infty}^{\infty} u(i)h(k-i) \tag{11.16}$$

If the system is causal, for example if $h(k)=0, \quad for \quad k<0$, then relation(11.16) becomes

$$y(k) = \sum_{i=0}^{\infty} u(i)h(k-i) = \sum_{i=0}^{\infty} u(k-i)h(i) \tag{11.17}$$

If both the system and the input signal are causal, i.e., if $h(k)=0$ and $u(k)=0$ for $k<0$, then equation(11.17) becomes

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i) = \sum_{i=0}^{k} u(k-i)h(i) \tag{11.18}$$

Then the output value $y(0)$, $y(1)$, $\cdots$, $y(k)$ can be acquired from equation(11.18) as follows:

$$\begin{gathered} y(0)=h(0)u(0) \\ y(1)=h(0)u(1)+h(1)u(0) \\ y(2)=h(0)u(2)+h(1)u(1)+h(2)u(0) \\ \cdots \\ y(k)=h(0)u(k)+h(1)u(k-1)+\cdots+h(k)u(0) \end{gathered}$$

Or written compactly as

$$y = Hu = Uh \tag{11.19}$$

Here,

$$y=\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k) \end{bmatrix} \qquad u=\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k) \end{bmatrix} \qquad h=\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(k) \end{bmatrix}$$

$$H=\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & & \\ \vdots & \vdots & \ddots & \\ h(k) & h(k-1) & \cdots & h(0) \end{bmatrix} \quad and \quad U=\begin{bmatrix} u(0) & 0 & \cdots & 0 \\ u(1) & u(0) & & \\ \vdots & \vdots & \ddots & \\ u(k) & u(k-1) & \cdots & u(0) \end{bmatrix}$$

Equation(11.19) can be used to determine the impulse response $h(k)$ if the input $u(k)$ and the output $y(k)$ are given. Then we have

$$h = U^{-1}y, \quad if \quad u(0) \neq 0$$

Above procedure is deconvolution procedure in fact since it is the reverse of convolution. And it constitutes a simple identification method for a discrete-time system.

4. Analysis of state equation

Solution of state equation can be given in chapter3 for linear time-invariant homogeneous system. For common difference equation, the solution form is similar to that in chapter3. From equation(11.11a), we can get the difference equation with constant coefficient matrices:

$$x(k+1) = Ax(k) + Bu(k) \tag{11.20}$$

Then, we have

For $k=0$: $x(1) = Ax(0) + Bu(0)$

For $k=1$:

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) = A[Ax(0) + Bu(0)] + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1) = A^2x(0) + \sum_{i=0}^{1} A^{1-i}Bu(i) \end{aligned}$$

For $k=2$:

$$x(3) = Ax(2) + Bu(2) = A\left[A^2x(0) + ABu(0) + Bu(1)\right] + Bu(2)$$
$$= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) = A^3x(0) + \sum_{i=0}^{2} A^{2-i}Bu(i)$$

For $k = 3$:

$$x(4) = Ax(3) + Bu(3) = A\left[A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)\right] + Bu(3)$$
$$= A^4x(0) + A^3Bu(0) + A^2B(1) + ABu(2) + Bu(3) = A^4x(0) + \sum_{i=0}^{3} A^{3-i}Bu(i)$$

Similarly, we can write out the following general expression for $k = 4, \quad 5, \quad 6, \quad \cdots$:

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i) \tag{11.21}$$

Proof

Mathematical induction is here utilized to demonstrate above deducing conclusion.

Postulate there are system state vector $x(0)$ and input vector $u(0)$ under initial condition.

When $k = 1$, from equation(11.21) we have the following expression:

$$x(2) = A^2x(0) + \sum_{i=0}^{1} A^{1-i}Bu(i) = A^2x(0) + ABu(0) + Bu(1)$$

Then from equation(11.20) we may calculate out the following expression for $k = 1$:

$$x(2) = Ax(1) + Bu(1) = A\left[Ax(0) + Bu(0)\right] + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$
$$= A^2x(0) + \sum_{i=0}^{1} A^{1-i}Bu(i)$$

Hence, when $k = 1$, the expression from equation(11.21) is correct.

Then, we postulate that when $k = k$, there is also the following form expression which is also correct.

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i)$$

When $k = k + 1$, from equation(11.20) we can write out

$$x(k+1) = Ax(k) + Bu(k)$$

$x(k)$ is substituted into above expression, and get

$$x(k+1) = A\left[A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i)\right] + Bu(k) = A^{k+1}x(0) + \left[\sum_{i=0}^{k-1} A^{k-i}Bu(i) + Bu(k)\right]$$
$$= A^{k+1}x(0) + \left[A^k Bu(0) + A^{k-1}Bu(1) + \cdots + A^{k-i}Bu(i) + \cdots + ABu(k-1) + Bu(k)\right]$$
$$= A^{k+1}x(0) + \sum_{i=0}^{k} A^{k-i}Bu(i) = A^{k+1}x(0) + \sum_{i=0}^{(k+1)-1} A^{(k+1)-i-1}Bu(i)$$

From equation(11.21), we may get the following expression when $k = k + 1$

$$x(k+1) = A^{k+1}x(0) + \sum_{i=0}^{(k+1)-1} A^{(k+1)-i-1}Bu(i)$$

Hence, when $k = k+1$, expression(11.21) is always found; namely, the deducing conclusion(11.21) is correct.

According to equation(11.11), we may write out

$$y(k) = Cx(k) + Du(k) = CA^k x(0) + C\sum_{i=0}^{k-1} A^{k-i-1}Bu(i) + Du(k) \tag{11.22}$$

In equations(11.21) and (11.22), the matrix A^k is state transition matrix which has been narrated in chapter3, and it is denoted as follows:

$$\phi(k) = A^k \tag{11.23}$$

The matrix $\phi(k)$ is analogous to the matrix $\phi(t)$ of continuous-time system.

The state vector $x(k)$ can be also calculated from equation(11.11a) using the Z-transform. Take the Z-transform of both sides of the equation(11.11a) to yield

$$zX(z) - zx(0) = AX(z) + BU(z)$$

Or

$$X(z) = (zE_{unit} - A)^{-1} BU(z) + z(zE_{unit} - A)^{-1} x(0)$$

Taking the inverse Z-transform for above equation, we have

$$x(k) = A^k x(0) + A^{k-1}Bu(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i) \tag{11.24}$$

Equation(11.24) is the same as equation(11.21), as expected. It is evident that the state transition matrix can be expressed as

$$\phi(k) = A^k = Z^{-1}\left[z(zE_{unit} - A)^{-1}\right] \tag{11.25}$$

When the initial conditions hold for $k = k_0$, the above similar results have the following general forms:

$$\phi(k,\quad k_0) = A^{k-k_0} \tag{11.26a}$$

$$x(k) = A^{k-k_0}x(k_0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i) = \phi(k-k_0)x(k_0) + \sum_{i=0}^{k-1}\phi(k-i-1)Bu(i) \tag{11.26b}$$

$$\begin{aligned} y(k) &= CA^{k-k_0}x(k_0) + C\sum_{i=0}^{k-1} A^{k-i-1}Bu(i) + Du(k) \\ &= C\phi(k-k_0)x(k_0) + C\sum_{i=0}^{k-1}\phi(k-i-1)Bu(i) + Du(k) \end{aligned} \tag{11.26c}$$

Example11.2 For a discrete -time system, please find the state and the output vectors with zero initial condition, related matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution

In terms of equation(11.25), we have

$$\phi(z) = Z[\phi(k)] = z(zE_{unit} - A)^{-1} = \frac{1}{z^2 - 3z + 2}\begin{bmatrix} z(z-3) & z \\ -2z & z^2 \end{bmatrix}$$

Hence

$$\phi(k)=\begin{bmatrix} Z^{-1}\left[\dfrac{z(z-3)}{z^2-3z+2}\right] & Z^{-1}\left[\dfrac{z}{z^2-3z+2}\right] \\ Z^{-1}\left[\dfrac{-2z}{z^2-3z+2}\right] & Z^{-1}\left[\dfrac{z^2}{z^2-3z+2}\right] \end{bmatrix}$$

These results can be checked as follows. Since $\phi(k)=A^k$, it follows that $\phi(0)=E_{unit}$ and $\phi(1)=A$. Moreover, from the initial value theorem it follows that

$$\phi(0)=\lim_{z\to\infty}\phi(z)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=E_{unit}$$

Since the initial conditions are zero, the state vector can be calculated as follows:

$$X(z)=(zE_{unit}-A)^{-1}BU(z)=\frac{\phi(z)}{z}BU(z)=\frac{1}{(z-1)^2(z-2)}\begin{bmatrix}z\\ z^2\end{bmatrix}$$

Using the inverse Z-transform for above expression, we can obtain

$$x(k)=Z^{-1}[X(z)]=\begin{bmatrix}2^k-k-1\\ 2(2^k-1)-k\end{bmatrix}$$

From this expression, it follows that $x(0)=0$. Finally, the output of the system is given by the following expression:

$$y(k)=C^Tx(k)=2^k-k-1+2(2^k-1)-k=3\times 2^k-2k-3$$

11.2.3 Description of sampled-data systems

1.Sampling hold circuit

In modern control system, a continuous-time system is controlled using a digital computer shown in figure11.1. As a result, the closed-loop system involves continuous-time, as well as discrete-time, subsystems. To have a common base for the closed-loop system, it is logical to use the same mathematical model for both continuous-time and discrete-time systems. Difference equation may be used in such mathematical model, which can unify the study of closed-loop systems, since the study models and results such as transfer function, stability criteria, controller design and so on can be extended to cover the case of discrete-time systems.

The practical problem which we often come across in such systems is that the output of a discrete-time system, which is a discrete-time signal, may be the input to a continuous-time system. And vice versa, the output of a continuous-time system, which is a continuous-time signal, could be the input to a discrete-time system. This problem is dealt with using converters. There are two types of converters: D/A converters, which convert the discrete-time signals to analog or continuous-time signals, and A/D converters, which convert analog or continuous-time signals to discrete-time signals(see section2.3.7). The constant T is the sampling period which is up to sampling frequency. It is noted that before the discrete-time signal $y(kT)$ of the A/D converter is fed into a digital computer, it is first converted to a digital signal with the help of a device called a quantizer(see figure11.3). The digital signal is a sequence of 0 and 1 digits(see figure11.1).

In fact, sampling(A/D conversion) and desampling (D/A conversion) courses are simply narrated in section1.3.2. Here, we will further demonstrate D/A conversion in math.

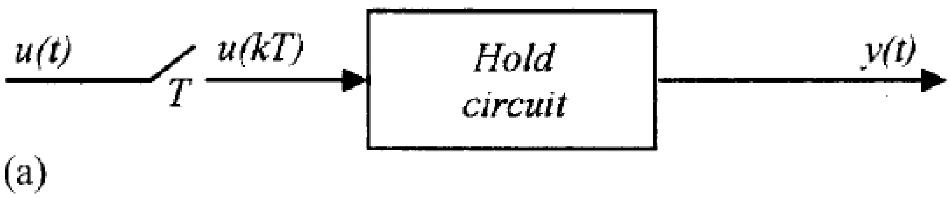
A D/A converter is actually a hold circuit which output $y(t)$ is a piecewise constant function in figure11.1. Specifically, the operation of the hold circuit is described by the following equations:

$$y(t)=u(kT), \quad for \quad kT \le t \le (k+1)T \tag{11.27a}$$

or

$$y(kT+\xi)=u(kT), \quad for \quad 0 \le \xi \le T \tag{11.27b}$$

In fact, the operation of a hold circuit, described by equation(11.27), can be equivalently described (from a mathematical viewpoint) by the idealized hold circuit shown in figure11.7.



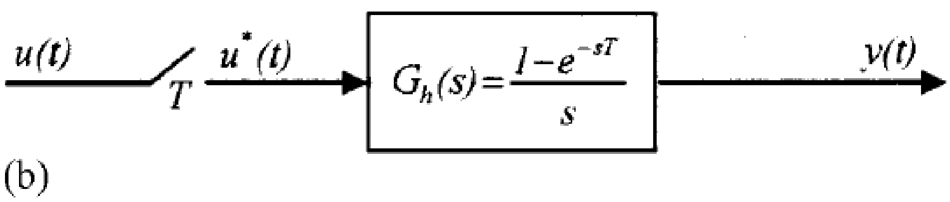


Figure11.7 Hold circuit block diagrams:(a) hold circuit; (b) idealized hold circuit

Here, we assume that the signal $u(t)$ goes through an ideal sampler δ_T , which output $u^*(t)$ is given by the following relation:

$$u^*(t)=\sum_{k=0}^{\infty}u(kT)\delta(t-kT) \tag{11.28}$$

Obviously, the sampled signal $u^*(t)$ is a sequence of impulse function. Applying Laplace transform into equation(11.28), we then can get

$$U^*(s)=\sum_{k=0}^{\infty}u(kT)e^{-skT} \tag{11.29}$$

Therefore, the output $Y(s)$ can be gained as

$$Y(s)=G_h(s)\bullet U^*(s)=\left(\frac{1-e^{-sT}}{s}\right)\sum_{k=0}^{\infty}u(kT)e^{-skT} \qquad (11.30)$$

We would have arrived at the same results as in equation(11.30) if we had calculated $Y(s)$ of the hold circuit shown in figure11.7(a). Since the impulse response $h(t)$ of the hold circuit is a gate function, i.e., $h(t)=\delta(t)-\delta(t-T)$, the output $y(t)$ should be the convolution of $u(kT)$ and $h(t)$, i.e.,

$$y(t)=u(kT)*h(t)=\sum_{k=0}^{\infty}u(kT)[\delta(t-kT)-\delta(t-kT-T)] \qquad (11.31)$$

Taking Laplace transform of equation(11.31), we can get

$$Y(s)=\sum_{k=0}^{\infty}u(kT)\left[\frac{e^{-skT}-e^{-s(k+1)T}}{s}\right] \qquad (11.32)$$

Equations(11.32) and (11.29) are identical. There, both configurations in figure11.7(a) and (b) are equivalent with regard to the output $y(t)$. It is clear that the device shown in figure11.7b can not be realized in practice and it is used only because it facilitates the mathematical description of the hold circuit.

2. Closed-loop feedback sampled-data system

As we have known, if the z-transform functions of the output sampled signal $y(k)$ and input sample signal $r(k)$ are $Y(z)$ and $R(z)$ respectively, the transfer function in the z-domain is

$$G(z)=\frac{Y(z)}{R(z)} \qquad (11.33)$$

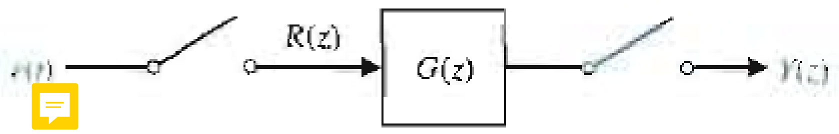


Figure11.8 System with sample output

If there are two samplers before and after a transfer function shown in figure11.8, we can determine the sampled output. And we can use an output sampler to depict the condition. We assume that both samplers have the sampling period and operate synchronously. Then the following transfer function is required.

$$Y(z)=G(z)R(z) \qquad (11.34)$$

Then we can represent equation(11.34), which is a z-transform equation using the block diagram in figure11.9.

Figure11.9 The z-transform transfer function in block diagram form

We consider closed-loop, sampled-data control systems. Consider the system shown in figure11.10(a). The sampled-data z-transform model of this figure with a sampled-output signal $Y(z)$ is shown in figure11.10(b). The closed-loop transfer function(using block diagram reduction) is

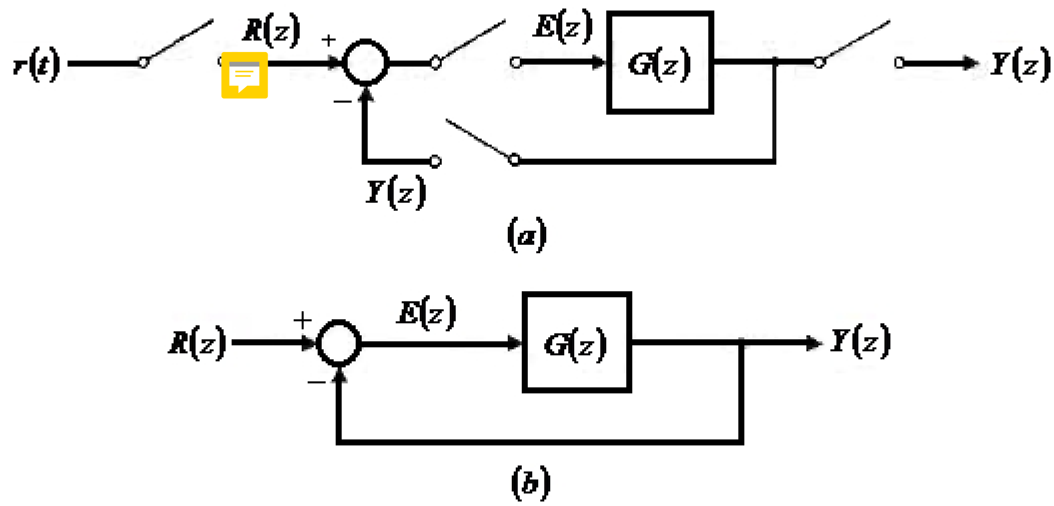


Figure11.10 Closed-loop feedback sampled-data systems

$$T(z)=\frac{Y(z)}{R(z)}=\frac{G(z)}{1+G(z)} \tag{11.35}$$

Here, we assume that the $G(z)$ is the z -transform of $G(s)=G_h(s)G_p(s)$, where $G_h(s)$ is the zero-order hold, and $G_p(s)$ is the process transfer function.

Now let's consider the block diagram of closed-loop system in figure11.11(a) and (b).

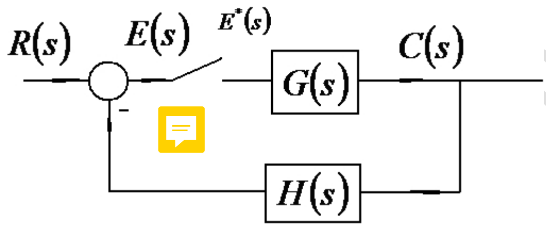


Figure11.11(a) System with error sampler

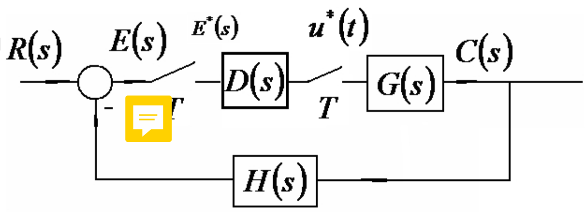


Figure11.11(b) System with two samplers

For the block diagram in figure11.11(a), we can deduce the error transfer function. From figure11.11(a), we can write out the following transfer functions:

$$C(s)=G(s)E^*(s)$$
$$E(s)=R(s)-H(s)C(s)$$

Merge above two equations to gain

$$E(s)=R(s)-H(s)G(s)E^*(s) \tag{11.36}$$

Sample equation(11.35) and achieve

$$E^*(s)=R^*(s)-[H(s)G(s)]^*E^*(s),\quad C^*(s)=G^*(s)E^*(s)$$

namely, $E^*(s)=\dfrac{R^*(s)}{1+[H(s)G(s)]^*}$, $\dfrac{C^*(s)}{R^*(s)}=\dfrac{G^*(s)}{1+[H(s)G(s)]^*}$

Hence, the error transfer function and closed-loop sampled transfer function are

$$\frac{E^*(s)}{R^*(s)}=\frac{1}{1+[H(s)G(s)]^*}\quad \text{and}\quad \frac{C^*(s)}{R^*(s)}=\frac{G^*(s)}{1+[H(s)G(s)]^*} \tag{11.37}$$

Their z -transform functions are

$$E(z)=\frac{R(z)}{1+GH(z)}, \quad C(z)=\frac{R(z)G(z)}{1+GH(z)}$$

Namely,

$$\phi(z)=\frac{C(z)}{R(z)}=\frac{G(z)}{1+GH(z)}, \quad \phi_e(z)=\frac{E(z)}{R(z)}=\frac{1}{1+GH(z)} \tag{11.38}$$

Similarly, for figure11.11(b), the following equations can be gained:

$$E(s)=R(s)-H(s)C(s)$$
$$C(s)=G(s)U^*(s)$$
$$U(s)=D(s)E^*(s)$$

Then, sample above equations and then we have

$$E^*(s)=R^*(s)-[H(s)G(s)]^*U^*(s), \quad U^*(s)=D^*(s)E^*(s), \quad C^*(s)=G^*(s)U^*(s)$$

Adjust and merge above sampled functions, then we have

$$E^*(s)=\frac{R^*(s)}{1+[H(s)G(s)]^*D^*(s)}, \quad C^*(s)=\frac{G^*(s)D^*(s)R^*(s)}{1+[H(s)G(s)]^*D^*(s)}$$

Therefore, the closed-loop z -transform functions are

$$\phi(z)=\frac{C(z)}{R(z)}=\frac{G(z)D(z)}{1+GH(z)D(z)}, \quad \phi_e(z)=\frac{E(z)}{R(z)}=\frac{1}{1+GH(z)D(z)} \tag{11.39}$$

3. Converting methods from $G(s)$ to $G(z)$

In order to convert a continuous-time transfer function $G(s)$ to the transfer function $G(z)$ of corresponding discrete-time system, various techniques have been proposed. Special method from $G(s)$ to $G(z)$ has been narrated in section2.3.7. In the following, we will further present some of the most popular conversion techniques.

(1) The backward difference method

For simplicity, consider the simple case of a first-order system described by the transfer function:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{a}{s+a} \tag{11.40}$$

According to equation(11.40), we may get its differential equation with zero initial condition.

$$y'+ay=au \tag{11.41a}$$

Integrating both sides of the differential equation from 0 to t, we may obtain:

$$\int_0^t \frac{dy}{dt}dt+a\int_0^t ydt=a\int_0^t udt \tag{11.41b}$$

We suppose that the sampling value of output value $y(t)$ at any sampling instants, i.e., at the points where $t=kT$, can be gained. Then the integral equation above can become

$$\int_0^{kT} \frac{dy}{dt} dt + a\int_0^{kT} y dt = a\int_0^{kT} u dt \tag{11.41c}$$

Here, T is sampling period constant.

Hence, we may acquire

$$y(kT) - y(0) = -a\int_0^{kT} y dt + a\int_0^{kT} u dt \tag{11.41d}$$

Replacing kT by $(k-1)T$ from equation(11.37), we may get

$$y[(k-1)T] - y(0) = -a\int_0^{(k-1)T} y dt + a\int_0^{(k-1)T} u dt \tag{11.41e}$$

Subtracting equation(11.38) from equation(11.37), we further obtain

$$y(kT) - y[(k-1)T] = -a\int_{(k-1)T}^{kT} y dt + a\int_{(k-1)T}^{kT} u dt \tag{11.41f}$$

Both terms on the right-hand side of equation(11.39) may be calculated approximately in various ways. If approximation is realized by applying backward difference method, equation(11.39) can become the following form:

$$y(kT) - y[(k-1)T] = -aTy(kT) + aTu(kT)$$

Namely,

$$y(kT) = y[(k-1)T] - aT[y(kT) - u(kT)] \tag{11.42a}$$

Apparently, equation(11.40) is an equivalent difference equation from the differential equation(11.34). To get $G(z)$, we only need to take the Z-transform for equation(11.40) to yield:

$$Y(z) = z^{-1}Y(z) - aTY(z) + aTU(z)$$

Namely,

$$G(z) = \frac{Y(z)}{U(z)} = \frac{aT}{1 - z^{-1} + aT} = \frac{a}{\left[\frac{1-z^{-1}}{T}\right] + a} \tag{11.42b}$$

Extending the results of above first-order transfer function to general case, we may acquire the following equivality for discretizing $G(s)$ using the backward difference method:

$$G(z) = G(s)\Big|_{s=\left[\frac{1-z^{-1}}{T}\right]} \tag{11.43}$$

(2) The forward difference method

In this case, the approximation of the two terms of right-hand side of equation(11.39) is done by the forward difference method. Working in the same way as in the previous case, we arrive at the following result for the discretizing $G(s)$ using the forward difference method:

$$y(kT) - y[(k-1)T] = -aTy[(k-1)T] + aTu[(k-1)T] \tag{11.44a}$$

$$Y(z) - z^{-1}Y(z) = -aTz^{-1}Y(z) + aTz^{-1}U(z) \tag{11.44b}$$

$$G(z)=\frac{Y(z)}{U(z)}=\frac{aTz^{-1}}{1-z^{-1}+aTz^{-1}}=\frac{a}{\frac{1-z^{-1}}{Tz^{-1}}+a} \tag{11.44c}$$

Samely, also extending the results of above first-order transfer function to general case, we may acquire the following equivality for discretizing $G(s)$ using the forward difference method:

$$G(z)=G(s)\big|_{s=\left(\frac{1-z^{-1}}{Tz^{-1}}\right)} \tag{11.45}$$

(3) The bilinear transform method

The Tustin transformation(also called bilinear transformation) is deduced from trapezoidal integration method. Such substitution may guarantee the stability of $G(z)$ and has a certain simulation accuracy. The trapezoidal integration formula given is as follows:

$$y(k+1)=y(k)+\frac{T}{2}\left(\dot{y}_k+\ddot{y}_{k+1}\right) \tag{11.46a}$$

Taking the Z-transform of equation(11.46a), we may achieve

$$(z-1)y=\frac{T}{2}(z+1)\dot{y} \tag{11.46b}$$

Then, applying Laplace transform on equation(11.46b), we can get

$$s=\frac{2}{T}\times\frac{z-1}{z+1} \tag{11.47}$$

Hence, the converting relation from $G(s)$ to $G(z)$ can be confirmed:

$$G(z)=G(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \tag{11.48}$$

(4) The invariant impulse response method

In this case, both $G(s)$ and $G(z)$ present a common characteristic in that their respective impulse functions $g(t)$ and $g(kT)$ are equal for $t=kT$. The following relation is achieved when

$$G(z)=Z[g(kT)],\quad where\quad g(kT)=L^{-1}[G(s)]\big|_{t=kT} \tag{11.49}$$

(5) The invariant step response method

In this case, both $G(s)$ and $G(z)$ present a common characteristic in that their respective step responses,i.e., the response $y(t)$ produced by the excitation $u(t)=1$ and the response $y(kT)$ produced by the excitation $u(kT)=1$, are equal for $t=kT$. This is achieved when

$$Z^{-1}\left[G(z)\frac{1}{1-z^{-1}}\right]=\left[L^{-1}\left[G(s)\frac{1}{s}\right]\right]_{t=kT} \tag{11.50}$$

Obviously, the left-hand side of equation(11.50) is equal to $y(kT)$, whereas the right-hand side is equal to $y(t)$ at $t=kT$. Applying the Z-transform on equation(11.50), we obtain

$$G(z) = \left(1 - z^{-1}\right) Z\left[\frac{G(s)}{s}\right] = Z\left[\frac{1 - e^{-Ts}}{s} G(s)\right] = Z\left[G_h(s) G(s)\right] \tag{11.51}$$

Here, $G_h(s)$ is the Laplace transfer function of zero-order hold circuit seen in section2.3.

(6) Pole-zero matching method

Now we consider the following common pole-zero transfer function in s-domain:

$$G(s) = K_s \frac{(s + \mu_1)(s + \mu_2)\cdots(s + \mu_m)}{(s + \pi_1)(s + \pi_2)\cdots(s + \pi_n)}, \quad m \le n \tag{11.52}$$

Then, assume that $G(z)$ has the following general form

$$G(z) = K_z \frac{(z + 1)^{n-m}(z + z_1)(z + z_2)\cdots(z + z_m)}{(z + p_1)(z + p_2)\cdots(z + p_n)} \tag{11.53}$$

Here, constants z_i and p_i are matched to the respective μ_i and π_i according to the following relation:

$$z_i = -e^{-\mu_i T} \quad \text{and} \quad p_i = -e^{-\pi_i T} \tag{11.54a}$$

The idea of zero-pole matching method is that the map $z = e^{sT}$ could reasonably be applied to the zeros and poles. This method consists of a set of heuristic rules for locating the zeros and poles and setting the gain of a Z-transform that will describe a discrete, equivalent transfer function that approximates the given $G(s)$. The matching rules are given as follows:

① All poles of $G(s)$ are mapped according to $z = e^{sT}$. If $G(s)$ has a pole at $s = -a$, then $G(z)$ has a pole at $z = e^{-aT}$. If $G(s)$ has a pole at $-a + jb$, then $G(z)$ has a pole at $re^{j\theta}$, where $r = e^{-aT}$ and $\theta = bT$.

②All finite zeros are also mapped by $z = e^{sT}$. If $G(s)$ has a zero at $s = -a$, then $G(z)$ has a zero at $z = e^{-aT}$, and so on.

③ The zeros of $G(s)$ at $s = \infty$ are mapped in $G(z)$ to the point $z = -1$. The rationale behind this rule is that the map of real frequencies from $j\omega = 0$ to increasing ω is onto the unit circle at $z = e^{j0} = 1$ until $z = e^{j\pi} = -1$. Thus the point $z = -1$ represents the highest frequency possible in the discrete transfer function in real way, so it is approximate that if $G(s)$ is zero at the highest (continuous) frequency, $|G(z)|$ should be zero at $z = -1$, the highest frequency can be processed by the digital filter.

(a) if no delay in the discrete response is desired, all zeros at $s = \infty$ are mapped to $z = -1$.

(b) If one sample period delay is desired to give the computer time to complete the output calculation, then one of the zeros at $s = \infty$ is mapped to $z = \infty$ and the others mapped to $z = -1$. With this choice, $G(z)$ is left with a number of finite zeros one fewer than the number of finite poles.

④The gain of the digital filter is selected to match the gain of $G(s)$ are the band center or a similar critical point. In most control applications, the critical frequency is $s=0$, and hence we typically select the gain so that

$$G(s)\big|_{s=0}=G(z)\big|_{z=1} \quad \text{(gain matching)} \qquad (11.54b)$$

The $n-m$ multiple zeros $(z+1)^{n-m}$ which appear in $G(z)$ represent the order difference between the numerator's polynomial and the denominator's polynomial in equation(11.52). The constant K_z is calculated so as to satisfy particular requirements. For example, when we are interested in the behavior of a system at low frequencies(and this is the usual case in control system), K_z is chosen such that $G(z)$ and $G(s)$ are equal for $z=1$ and $s=0$, respectively, i.e., the following relation should be found:

$$G(z)\big|_{z=1}=K_z\frac{2^{n-m}(1+z_1)(1+z_2)\cdots(1+z_m)}{(1+p_1)(1+p_2)\cdots(1+p_n)}=G(s)\big|_{s=0}=K_s\frac{\mu_1\mu_2\cdots\mu_m}{\pi_1\pi_2\cdots\pi_n} \qquad (11.55)$$

Example11.3 For the system with transfer function $G(s)=a/(s+a)$, please find all above six equivalent descriptions of $G(z)$.

Solution

After the several simple algebraic calculation in terms of above 6 conversion methods, all six descriptions of $G(z)$ are found and summarized in table11.1.

Table11.1 Equivalent discrete-time transfer function $G(z)$ of continuous-time transfer function $G(s)=a/(s+a)$

Conversion method	Conversion from $G(s)$ to $G(z)$	Equivalent discrete-time transfer function $G(z)$
Backward difference method	$s=\frac{1-z^{-1}}{T}$	$G(z)=\frac{a}{\left[\frac{1-z^{-1}}{T}\right]+a}$
Forward difference method	$s=\frac{1-z^{-1}}{Tz^{-1}}$	$G(z)=\frac{a}{\left[\frac{1-z^{-1}}{Tz^{-1}}\right]+a}$
Bilinear transform method	$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$	$G(z)=\frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$
Invariant impulse response method	$G(z)=Z[g(t)]$ where $g(t)=Z^{-1}[G(s)]$	$G(z)=\frac{a}{1-e^{-At}z^{-1}}$
Invariant step response method	$G(z)=Z\left[\frac{1-e^{Ts}}{s}G(s)\right]$	$G(z)=\frac{(1-e^{-aT})z^{-1}}{1-e^{-aT}z^{-1}}$

Pole-zero matching method	$G(z)=K_z\left[\dfrac{z+1}{z+p_1}\right]$	$G(z)=\left(\dfrac{1-e^{-aT}}{2}\right)\left(\dfrac{1+z^{-1}}{1-e^{-aT}z^{-1}}\right)$

Example11.4 Consider a second-order continuous-time system having the following transfer function:

$$H(s)=\frac{b}{s(s+a)}$$

Please find $H(z)$ using the invariant impulse response method.

Solution

$$h(t)=L^{-1}[H(s)]=\frac{b}{a}L^{-1}\left[\frac{1}{s}-\frac{1}{s+a}\right]=\frac{b}{a}\left(1-e^{-at}\right)$$

Hence, the descritized function $h(kT)$ can be achieved

$$h(kT)=h(t)\big|_{t=kT}=\frac{b}{a}\left(1-e^{-akT}\right)$$

The Z-transform function $H(z)$ of $h(kT)$ is the following:

$$H(z)=Z[h(kT)]=\frac{\frac{b}{a}}{1-z^{-1}}-\frac{\frac{b}{a}}{1-e^{-aT}z^{-1}}=\frac{bz^{-1}\left(1-e^{-aT}\right)}{a\left(1-z^{-1}\right)\left(1-e^{-aT}z^{-1}\right)}$$

3. Converting methods from differential state space equation to difference state equation

Now let's together consider the continuous-time multiple-input-multiple-input (MIMO) open-loop system shown in figure11.9.

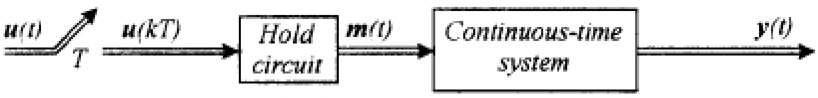


Figure11.9 MIMO open-loop system with a sampler and hold circuit

The system in figure11.9 may be described in the state space by the following equations:

$$\dot{x}(t)=Fx(t)+Gm(t) \tag{11.56a}$$

$$y(t)=Cx(t)+Dm(t) \tag{11.56b}$$

In the following, equation group (11.56) will be approximately changed into difference form. To this end, consider the piecewise constant excitation vector $m(t)$ described by

$$m(t)=u(kT),\quad for\quad kT\le t<(k+1)T \tag{11.57}$$

Then, solving equation(11.56a) for $x(t)$, we may get

$$x(t)=e^{Ft}x(0)+\int_0^t e^{F(t-\tau)}Gm(\tau)d\tau \tag{11.58}$$

In fact, above similar solution has been given in section3.2.Here, we focus on how to change above equation into difference equation. According to equation(11.57), $m(0)=u(0)$ for $0\le t\le T$, and hence equation(11.58) becomes

$$x(t)=e^{Ft}x(0)+\int_0^t e^{F(t-\tau)}Gu(0)d\tau \tag{11.59}$$

The state vector $x(t)$, for $t=T$, will be

$$x(t)=e^{FT}x(0)+\int_0^T e^{F(T-\tau)}Gu(0)d\tau \tag{11.60}$$

Define

$$A(T)=e^{FT} \tag{11.61a}$$

$$B(T)=\int_0^T e^{F(T-\tau)}Gd\tau=\int_0^T e^{F\lambda}Gd\lambda,\quad \lambda=T-\tau \tag{11.61b}$$

Then, equation(11.59) for $t=T$ will be simplified as follows:

$$x(T)=A(T)x(0)+B(T)u(0) \tag{11.62}$$

Repeating above procedure for $T\le t<2T$, $2T\le t<3T$, $\cdots$ $kT\le t<(k+1)T$, we may get the following general form:

$$x[(k+1)T]=A(T)x(kT)+B(T)u(kT) \tag{11.63}$$

Hence, the output of equation(11.56b) can be written into the following form:

$$y(kT)=Cx(kT)+Du(kT) \tag{11.64}$$

Therefore, the state differential equations(11.56) can be changed into corresponding difference equations, as follows:

$$x[(k+1)T]=A(T)x(kT)+B(T)u(kT) \tag{11.65a}$$

$$y(kT)=Cx(kT)+Du(kT) \tag{11.65b}$$

The state equations(11.56) and (11.65) are equivalent only for the time instants $t=kT$ under the constraint that the input vectors $m(t)$ and $u(t)$ satisfy the condition(11.57).

The transfer function matrix of the continuous-time system(11.56) is

$$H(s)=C(sE_{unit}-F)^{-1}G+D \tag{11.66}$$

And the transfer function matrix of the equivalent discrete-time system(11.65) is

$$G(z)=C[zE_{unit}-A(T)]^{-1}B(T)+D \tag{11.67}$$

These matrices are related in terms of previous content in section11.2.3, as follows:

$$G(z)=Z[G_h(s)H(s)]=Z\left[\left(\frac{1-e^{-sT}}{s}\right)H(s)\right] \tag{11.68}$$

Here, To conveniently convert the state differential equation into equivalent difference equation, we need to determine the system matrix $A(T)$ and input matrix $B(T)$ using definition(11.61). Then the following relation cam be gained:

$$A(T)=e^{Ft}\Big|_{t=T}=\left[L^{-1}\left[(sE_{unit}-F)^{-1}\right]\right]_{t=T} \tag{11.69a}$$

$$B(T)=\int_0^T\left[L^{-1}\left[(sE_{unit}-F)^{-1}\right]\right]_{t=\lambda}Gd\lambda=\int_0^T A(\lambda)Gd\lambda \tag{11.69b}$$

Example11.5 Consider the following continuous-time system:

$$\dot{x}(t)=Fx(t)+Gm(t),\quad y(t)=Cx(t)+Du(t)$$

Here

$$F=\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix},\quad G=\begin{bmatrix}2\\ 1\end{bmatrix},\quad C=\begin{bmatrix}0\\ 1\end{bmatrix},\quad D=0$$

Please find the equivalent discrete-time system.

Solution

$$sE_{unit}-F=\begin{bmatrix}s+1 & 0\\ -1 & s\end{bmatrix},\quad (sE_{unit}-F)^{-1}=\begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{1}{s(s+1)} & \frac{1}{s}\end{bmatrix},\quad L^{-1}\left[(sE_{unit}-F)^{-1}\right]=\begin{bmatrix}e^{-t} & 0\\ 1-e^{-t} & 1\end{bmatrix}$$

Hence, from equation(11.69a), we can gain

$$A(T)=\left[L^{-1}\left[(sE_{unit}-F)^{-1}\right]\right]_{t=T}=\begin{bmatrix}e^{-T} & 0\\ 1-e^{-T} & 1\end{bmatrix}$$

Moreover, from equation(11.69b), we can gain

$$B(T)=\int_0^T e^{F\lambda}Gd\lambda=\int_0^T A(\lambda)Gd\lambda=\int_0^T\begin{bmatrix}e^{-\lambda} & 0\\ 1-e^{-\lambda} & 1\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}d\lambda=\int_0^T\begin{bmatrix}2e^{-\lambda}\\ 3-2e^{-\lambda}\end{bmatrix}d\lambda=\begin{bmatrix}2\left(1-e^{-T}\right)\\ 3T-2\left(1-e^{-T}\right)\end{bmatrix}$$

11.2.4 Analysis of sampled-data systems

1. Analysis based on discrete-time state equation

To solve equation(11.65), we may utilize the results of section11.2.3 since they differ only by the constant T (sampling period). Therefore, we get that the general solution of equation(11.65a) is given by

$$x(kT)=\phi[(k-k_0)T]x(k_0T)+\sum_{i=k_0}^{k-1}\phi[(k-i-1)T]B(T)u(iT) \qquad (11.70a)$$

and the general solution of equation(11.65b) by

$$y(kT)=C\phi[(k-k_0)T]x(k_0T)+C\sum_{i=k_0}^{k-1}\phi[(k-i-1)T]B(T)u(iT)+Du(kT) \qquad (11.70b)$$

where $\phi[(k-k_0)T]$ is the system state transition matrix, given by the following relation:

$$\phi[(k-k_0)T]=[A(T)]^{k-k_0}$$

Obviously, if set $T=1$ in equation(11.70), then we obtain the formulas(11.21) and (11.22), respectively.

2. Analysis based on $H(kT)$

Consider the continuous-time system shown in figure11.10, where the two samplers are synchronized. The output vector $y(kT)$,i.e., the vector $y(t)$ at the sampling points $t=kT$, is the following:

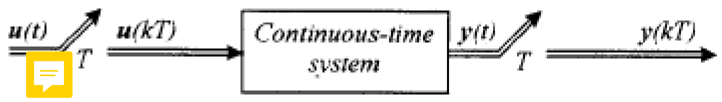


Figure11.10 A continuous-time system with input and output samplers

$$y(kT)=\sum_{i=0}^{\infty}H(kT-iT)u(iT) \qquad (11.71)$$

Equation(11.71) represents a convolution which has been known. If set $T=1$, then equation(11.71) is the vector form of the scalar convolution(11.17)

3. Analysis based on $H(z)$

If we execute the Z-transform of equation(11.71), we obtain the following expression for the output vector:

$$Y(z)=H(z)U(z) \qquad (11.72)$$

Here

$$Y(z)=Z[y(kT)],\quad U(z)=Z[u(kT)],\quad and \quad H(z)=Z[H(kT)] \qquad (11.73)$$

Example11.6 Consider the following transfer function of describing oscillator for $T=1,\quad \omega=\frac{\pi}{2}$.

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\omega^2}{s^2+\omega^2}$$

Please find

(1) continuous-time state space equation and output equation;

(2) matrices $A(T)$ and $B(T)$ of equivalently discrete-time state space equation;

(3) System state transition matrix $\phi(kT)$;

(4) State space vector $x(kT)$;

(5) The output vector $y(kT)$.

Solution

(1) from the given information, we know that the differential equation with zero initial condition is the following:

$$\ddot{y}+\omega^2 y(t)=\omega^2 u(t)$$

Select that the state variable is such

$$x_1=y$$
$$x_2=\omega^{-1}\dot{x}_1=\omega^{-1}\dot{y}$$
$$\omega^{-1}\ddot{y}=\dot{x}_2$$

Hence, we may write out the following state space expressions:

$$\dot{x}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & \omega\\ -\omega & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \omega\end{bmatrix}u(t),\qquad y(t)=\begin{bmatrix}1\\ 0\end{bmatrix}x(t)$$

Then, related matrices of system can be known:

$$F=\begin{bmatrix}0 & \omega\\ -\omega & 0\end{bmatrix},\quad G=\begin{bmatrix}0\\ \omega\end{bmatrix},\quad C=\begin{bmatrix}1\\ 0\end{bmatrix}$$

(2) find matrices $A(T)$ and $B(T)$.

$$(sE_{unit}-F)=\begin{bmatrix}s & -\omega\\ \omega & s\end{bmatrix},\qquad (sE_{unit}-F)^{-1}=\begin{bmatrix}\dfrac{s}{s^2+\omega^2} & \dfrac{\omega}{s^2+\omega^2}\\ \dfrac{-\omega}{s^2+\omega^2} & \dfrac{s}{s^2+\omega^2}\end{bmatrix},$$

$$L^{-1}\left[(sE_{unit}-F)^{-1}\right]=\begin{bmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{bmatrix}$$

Hence, in terms of equation(11.69), we can obtain

$$A(T)=\left[L^{-1}\left[(sE_{unit}-F)^{-1}\right]\right]_{t=T}=\begin{bmatrix}\cos\omega T & \sin\omega T\\ -\sin\omega T & \cos\omega T\end{bmatrix}$$

$$B(T)=\int_0^T e^{F\lambda}G\,d\lambda=\int_0^T\begin{bmatrix}\cos\omega\lambda & \sin\omega\lambda\\ -\sin\omega\lambda & \cos\omega\lambda\end{bmatrix}\begin{bmatrix}0\\ \omega\end{bmatrix}d\lambda=\int_0^T\begin{bmatrix}\omega\sin\omega\lambda\\ \omega\cos\omega\lambda\end{bmatrix}d\lambda=\begin{bmatrix}1-\cos\omega T\\ \sin\omega T\end{bmatrix}$$

(3) System state transition matrix $\phi(kT)$;

For the given values $T=1$, $\omega=\dfrac{\pi}{2}$, matrices A, B, C become

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\qquad B=\begin{bmatrix}1\\ 1\end{bmatrix},\qquad C=\begin{bmatrix}1\\ 0\end{bmatrix}$$

In terms of equation(11.25), we can gain

$$\phi(z)=Z[\phi(kT)]=z(zE_{unit}-A)^{-1}=\begin{bmatrix}\dfrac{z^2}{z^2+1} & \dfrac{z}{z^2+1}\\ \dfrac{-z}{z^2+1} & \dfrac{z^2}{z^2+1}\end{bmatrix}$$

Therefore, system state transition matrix $\phi(kT)$ may be obtained as

$$\phi(kT)=Z^{-1}[\phi(z)]=\begin{bmatrix}Z^{-1}\left[\dfrac{z^2}{z^2+1}\right] & Z^{-1}\left[\dfrac{z}{z^2+1}\right]\\ Z^{-1}\left[\dfrac{z}{z^2+1}\right] & Z^{-1}\left[\dfrac{z^2}{z^2+1}\right]\end{bmatrix}=\begin{bmatrix}\cos\dfrac{k\pi T}{2} & \sin\dfrac{k\pi T}{2}\\ -\sin\dfrac{k\pi T}{2} & \cos\dfrac{k\pi T}{2}\end{bmatrix}$$

(4) State space vector $x(kT)$;

$$X(z)=(zE_{unit}-A)^{-1}BU(z)=\begin{bmatrix}\dfrac{z}{z^2+1} & \dfrac{1}{z^2+1}\\ \dfrac{-z}{z^2+1} & \dfrac{z}{z^2+1}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\left[\dfrac{z}{z-1}\right]=\dfrac{1}{(z^2+1)(z-1)}\begin{bmatrix}z(z+1)\\ z(z-1)\end{bmatrix}$$

Hence,

$$x(kT)=Z^{-1}[X(z)]=\begin{bmatrix}1-\cos\dfrac{k\pi T}{2}\\ \sin\dfrac{k\pi T}{2}\end{bmatrix}$$

(5) System output vector $y(kT)$ is given by

$$y(kT)=C^{T}x(kT)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1-\cos\dfrac{k\pi T}{2}\\ \sin\dfrac{k\pi T}{2}\end{bmatrix}=1-\cos\frac{k\pi T}{2}$$

11.3 Stability

11.3.1 Definition of linear time-invariant discrete-time systems

1. Stability introduction of discrete-time system

Let's consider the following discrete-time system

$$x[(k+1)T]=f[x(kT),\quad kT,\quad u(kT)],\quad x(k_0T)=x_0 \qquad (11.74)$$

Let $u(kT)=0$, for $k\geq k_0$ .Moreover, let $x(kT)$ and $\tilde{x}(kT)$ be the solution of equation(11.74) when the initial conditions are $x(k_0T)$ and $\tilde{x}(k_0T)$, respectively. Also let the symbol $\|\bullet\|$ represent the Euclidean norm:

$$\|x\|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

In the following, we will focus on the definition and discussion of the stability of discrete-time system.

Definition11.3.1: Stability

The solution $x(kT)$ of equation(11.74) is stable if for a given $\varepsilon>0$ there exists a $\delta(\varepsilon,\ k_0)>0$ such that all solutions satisfying $\left\|x(k_0T)-\tilde{x}(k_0T)\right\|<\varepsilon$ imply that $\left\|x(k_0T)-\tilde{x}(k_0T)\right\|<\delta$ for all $k\geq k_0$.

Definition11.3.2: Asymptotic stability

The solution $x(kT)$ of equation(11.74) is asymptotically stable if it is stable and if $\left\|x(k_0T)-\tilde{x}(k_0T)\right\|\to 0$ as $k\to+\infty$, under the constraint that $\left\|x(k_0T)-\tilde{x}(k_0T)\right\|$ is sufficiently small.

In the case where the system(11.74) is stable in accordance with definition11.3.1, the point $x(k_0T)$ is called the *equilibrium point of system*. In the case where the system(11.74) is asymptotically stable, the equilibrium point is the origin point 0.

2. Stability introduction of discrete-time system

Now consider the following linear time-invariant discrete-time system

$$\left.\begin{aligned} x[(k+1)T] &= Ax(kT)+Bu(kT), \quad x(k_0T)=x_0 \\ y(kT) &= Cx(kT)+Du(kT) \end{aligned}\right\} \tag{11.75}$$

After applying definition11.3.1 to system(11.75), we have the following theorem.

Theorem11.3.1

System(11.75) is stable according to definition11.3.1 if , and only if, the eigenvalues λ_i of the matrix A, i.e., the roots of the characteristic equation $|\lambda E_{unit}-A|=0$, lie inside the unit circle(i.e., $|\lambda_i|<1$), or the matrix A has eigenvalues on the unit circle(i.e., $|\lambda_i|=1$) of multiplicity one.

After applying definition11.3.2 to system(11.75), we have the following theorem.

Theorem11.3.2

System(11.75) is asymptotically stable if, and only if, $\lim\limits_{k\to\infty} x(kT)=0$, for every $x(k_0T)$, when $u(kT)=0$ ($k \ge k_0$).

On the basis of theorem11.3.2, prove the following theorem.

Theorem11.3.3

System(11.75) is asymptotically stable if, and only if, the eigenvalues λ_i of A are inside the unit circle.

Demonstration

When $u(kT)=0$ ($k \ge k_0$)., the state vector $x(kT)$ is given by

$$x(kT)=A^{k-k_0}x(k_0T) \tag{11.76}$$

Postulate that eigenvalues λ_1, λ_2, $\cdots$, λ_n of the system matrix A are different. Then, according to the Sylvester theorem, the matrix A^k can be written as

$$A^k=\sum_{i=1}^{n} A_i\lambda_i^k \tag{11.77}$$

Here, A_i are special matrices which depend only on A. Substituting equation(11.77) into equation(11.76), we can obtain

$$x(kT)=\sum_{i=1}^{n} A_i\lambda_i^{k-k_0}x(k_0T) \tag{11.78}$$

Hence,

$$\lim_{k\to\infty} x(kT)=\left[\sum_{i=1}^{n} A_i x(k_0T)\right]\left[\lim_{k\to\infty}\lambda_i^{k-k_0}\right] \tag{11.79}$$

From equation(11.79), it is obvious that $\lim\limits_{k\to\infty} x(kT)=0$, $\forall x(k_0T)$ if, and only if, $\lim\limits_{k\to\infty}\lambda_i^{k-k_0}=0$, $\forall i=1,2,\cdots,n$, which is true if and only if $|\lambda_i|<1$, $\forall i=1,2,\cdots,n$, where $|\bullet|$

stands for the magnitude of a complex number.

A comparison between linear time-invariant continuous-time systems and linear time-invariant discrete-time systems with respect to the asymptotic stability is listed in table11.2.

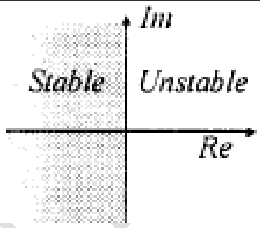
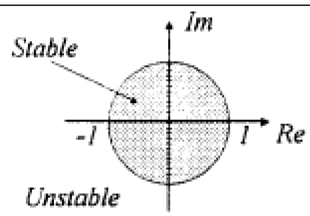
3. Bounded-input bounded-output stability

Definition11.3.3

A linear time-invariant system is bounded-input bounded-output(BIBO) stable if a bounded input produces a bounded output for every initial condition.

After applying definition11.3.3 to system(11.75), we have the following theorem.

Table11.2 Comparison of asymptotic stability between continuous-time and discrete-time systems

	Continuous-time system	Discrete-time system
State space equations	$\dot{x}(t)=Fx(t)+Gu(t)$ $y(t)=Cx(t)+Du(t)$	$x[(k+1)T]=Ax(kT)+Bu(kT)$ $y(kT)=Cx(kT)+Du(kT)$
Characteristic equation	$\lvert\lambda E_{unit}-F\rvert=0$	$\lvert\lambda E_{unit}-A\rvert=0$
Stability definition	$\mathrm{Re}\{\lambda_i\}<0$	$\lvert\lambda_i\rvert<1$
Stability description		

Theorem11.3.4

The linear time-invariant system(11.75) is BIBO stable if, and only if, the poles of the transfer function $H(z)=C(zE_{unit}-A)^{-1}B+D$, before any pole-zero cancellation, are inside the unit circle.

From definition11.3.3 we may conclude that asymptotic stability is the strongest, since it implies both stability and BIBO stability. It is easily known that stability does not imply BIBO stability and vice versa.

11.3.2 Stability criteria and application examples

The concept of stability has been presented in some depth in chapter4. For testing stability, various methods have been proposed such as Lyapunov criterion. The most popular methods for determining the stability of a discrete-time system are the following:

1. The Routh criterion;
2. The Jury criterion;
3. The Lyapunov method;
4. The root locus method;

5. The bode and Nyquist criteria.

In fact, above given methods have been completely narrated and explained in classical control theories and the chapter4 in this book. Here, we only briefly present criteria1 and 2 again.

1. Routh criterion for the discrete-time system

Consider the following polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \qquad (11.80)$$

The roots of the polynomial are the roots of the equation:

$$a(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0 \qquad (11.81)$$

As stated in theorem11.3.3, asymptotic stability is secured if all the roots of the characteristic polynomial lie inside the unit circle. The well-known Routh criterion for continuous-time systems is a simple method for determining if all the roots of an arbitrary polynomial lie in the left complex plane without requiring the determination of the values of the roots. The following bilinear transformation maps the unit circle of the z-plane into the left w-plane.

$$w = \frac{z+1}{z-1} \qquad \text{or} \qquad z = \frac{w+1}{w-1} \qquad (11.82)$$

Consequently, if the transformation is applied to equation(11.81), then the Routh criterion may be applied, as in the case of continuous-time systems.

Example11.7 The characteristic polynomial $a(z)$ of a system is given by

$$a(z) = z^2 + 0.7z + 0.1 .$$

Please investigate the stability of the system.

Solution

Applying the transformation(11.82) to $a(z)$ yields

$$a(w) = \left[\frac{w+1}{w-1}\right]^2 + 0.7\left[\frac{w+1}{w-1}\right] + 0.1 = \frac{(w+1)^2 + 0.7(w^2-1) + 0.1(w-1)^2}{(w-1)^2}$$

$$= \frac{1.8w^2 + 1.8w + 0.4}{(w-1)^2}$$

The numerator of $a(w)$ is called the "auxiliary" characteristic polynomial to which the well-known Routh criterion will be applied. For the present example, the Routh array is

$$\begin{array}{c|cc} w^2 & 1.8 & 0.4 \\ w^1 & 1.8 & 0 \\ w^0 & 0.4 & 0 \end{array}$$

The coefficients of the first column have the same sign and, according to Routh's criterion, the system is stable. We can reach the same result if we factorize $a(z)$ into a product of terms, in which case $a(z) = (z+0.5)(z+0.2)$. The two roots of $a(z)$ are -0.5 and -0.2, which are both inside the unit circle and hence the system is stable.

2. Jury criterion

It is useful to construct criteria which can directly show whether a polynomial $a(z)$ has all its roots inside the unit circle instead of determining its eigenvalues. Such a criterion, which is equivalent to the Routh criterionfor continuous-time systems, has been put forward by Jury, Cohn and Schur. This criterion is usually called the Jury criterion and it is described in detail below.

The Jury table is firstly formed for the polynomial $a(z)$, provided in equation(11.80), as shown in table11.3. The first two rows of the table are the coefficients of the polynomial $a(z)$ presented in the forward and reverse order, respectively. The third row is formed by multiplying the second row by $\beta_n = a_n / a_0$ and subtracting the result from the first row. Note that the last element of the third row becomes zero. The fourth row is identical to the third row, but in reverse order. The above procedure is repeated until the $2n+1$ row is reached. The last row consists of only a single element.

Theorem11.3.5: The stability Jury criterion

For polynomial in equation(11.80), if $a_0 > 0$, then the polynomial $a(z)$ has all its roots inside the unit circle if , and only if all a_0^k , $k = 0,1,\cdots,n-1$, are positive. If all coefficients a_0^k are not zero, then the number of negative coefficients a_0^k is equal to the number of roots which lie outside the unit circle.

If all coefficients a_0^k , $k = 1,2,\cdots,n$ are positive, then it can be shown that the condition $a_0^0 > 0$ is equivalent to the following two conditions:

$$a(1) > 0 \tag{11.83a}$$

$$(-1)^n a(-1) > 0 \tag{11.83b}$$

Relations(11.83 a and b) present necessary conditions for stability and may therefore be used to check for stability, prior to construction of the Jury table shown in table11.3.

Table11.3 The Jury table

a_0	a_1	$\cdots$	a_{n-1}	a_n	
a_n	a_{n-1}	$\cdots$	a_1	a_0	$\beta_n = a_n / a_0$
a_0^{n-1}	a_1^{n-1}	$\cdots$	a_{n-1}^{n-1}		
a_{n-1}^{n-1}	a_{n-2}^{n-1}	$\cdots$	a_0^{n-1}		$\beta_{n-1} = a_{n-1}^{n-1} / a_0^{n-1}$
$\vdots$					
a_0^0					

where $a_i^{n-1} = a_i^k - \beta_k a_{k-1}^k$ and $\beta_k = a_k^k / a_0^k$

Example11.8 The characteristic polynomial $a(z)$ of a system is given by

$$a(z) = z^2 + a_1 z + a_2 .$$

Please investigate the stability of the system.

Solution

The Jury table is formed as shown in table11.4. All the roots of the characteristics polynomial are inside the unit circle if

$$1-a_2^2>0 \quad \text{and} \quad \left(\frac{1-a_2}{1+a_2}\right)\left[(1+a_2)^2-a_1^2\right]>0$$

which lead to the conditions $-1<a_2<1$, $a_2>-1+a_1$, and $a_2>-1-a_1$

Table11.4 The Jury table of example11.8

1	a_1	a_2	
a_2	a_1	1	$\beta_2=a_2$
$1-a_2^2$	$a_1(1-a_2)$		
$a_1(1-a_2)$	$1-a_2^2$		$\beta_1=\dfrac{a_1}{1+a_2}$
$1-a_2^2-\dfrac{a_1^2(1-a_2)}{1+a_2}$			

11.4 Controllability and Observability

11.4.1 Controllability

Simply speaking, controllability is a property of a system which is strongly related to the ability of the system to go from a given initial state to a desired final state within a finite time. This theme has been narrated in section5.2 of chapter5 mainly for linear continuous-time system.Here, further discuss this theme only for discrete-time system. Consider the following system

$$x(k+1)=Ax(k)+Bu(k), \quad x(0)=x_0 \tag{11.84a}$$

$$y(k)=Cx(k) \tag{11.84b}$$

The state $x(k)$ at time k is given by equation(11.21), which can be rewritten as follows:

$$x(k)=A^k x(0)+\begin{bmatrix}B & AB & A^2B & \cdots & A^{k-1}B\end{bmatrix}\begin{bmatrix}u(k-1)\\u(k-2)\\\vdots\\u(1)\\u(0)\end{bmatrix} \tag{11.85}$$

Definition11.4.1

Assume that $|A| \neq 0$. Then system represented by equation(11.84a) is controllable if it is possible to find a controllable sequence $\{u(0),\ u(1),\ \cdots,\ u(q-1)\}$ which permits system reach an arbitrary final state $x(q) = \xi \in R^n$ within finite time from any initial state $x(k_0 T)$.

According to definition11.4.1, equation(11.85) can takes on the following form:

$$\xi - A^q x(0) = \begin{bmatrix} B & AB & \cdots & A^{q-1}B \end{bmatrix} \begin{bmatrix} u(q-1) \\ u(q-2) \\ \vdots \\ u(0) \end{bmatrix} \tag{11.86}$$

This relation is an inhomogenous algebraic system of equations, having the control sequence $\{u(0),\ u(1),\ \cdots,\ u(q-1)\}$ and the parameter q which is unknown. From linear algebra theory, it is known that this equation has a solution if and only if

$$rank\begin{bmatrix} B & AB & \cdots & A^{q-1}B & \xi - A^q x(0) \end{bmatrix} = rank\begin{bmatrix} B & AB & \cdots & A^{q-1}B \end{bmatrix}$$

This condition, for each arbitrary final state $x(q) = \xi$, holds true if and only if

$$rank\begin{bmatrix} B & AB & \cdots & A^{q-1}B \end{bmatrix} = n, \quad q \in N \tag{11.87}$$

Clearly, an increase in time q improves the possibility of satisfying condition(11.87). However, the Cayley-Hamilton theorem states that the terms $A^j B$, for $j \geq n$, are linearly dependent on the first n terms (i.e., on the terms $B,\ AB,\ \cdots,\ A^{n-1}B$). Thus, condition(11.87) holds true if and only if $q = n$, i.e., if

$$rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n \tag{11.88}$$

Therefore, we can gain the following theorem.

Theorem11.4.1

System in equation(11.84a) is controllable if and only if

$$rankS_c = rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n, \quad where \quad S_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{11.89}$$

Here the $n \times n$ judgment matrix S_c is called the controllability matrix.

Example11.9 A discrete-time system can be described by equation(11.84a), and related matrices are as follows

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Please find a input control sequence, if it exists, that can drive the system to the desired final state $\xi = [1 \quad 2]^T$.

Solution

Confirm the controllability matrix S_c and then judge whether given system is completely controllable.

$$S_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad rankS_c = 2$$

Obviously, the controllability matrix S_c is completely controllable, hence system may be driven to the desired final state. In terms of expression in equation(11.85), system state $x(2)$ can be confirmed as

$$x(2) = A^2x(0) + [B \quad AB]\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = [B \quad AB]\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} u(0) \\ u(0)+u(1) \end{bmatrix}$$

For $\xi = x(2)$, we may gain the corresponding equation group and get the solution $u(0)=1 \quad and \quad u(1)=1$. Hence the desired input control sequence is $\{u(0), \quad u(1)\} = \{1, \quad 1\}$.

11.4.2 Observability

Definition11.4.2

System described by equation group (11.84) is observable if there exists a finite time q such that the initial state $x(0)$ of the system may be uniquely determined on the basis of the input-output sequences $\{u(0), \quad u(1), \quad \cdots, \quad u(q-1)\}$ and $\{y(0), \quad y(1), \quad \cdots \quad y(q-1)\}$.

Now let's together consider the system(11.84). The influence of the input signal $u(k)$ on the behavior of the system can be always determined. Therefore, we postulate that $u(k)=0$ without loss of generality. We also assume that the output sequence $\{y(0), \quad y(1), \quad \cdots \quad y(q-1)\}$ is known for a certain q. Then in terms of equation(11.84b), we have the following equation group written using matrix form:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} \tag{11.90}$$

Here, use was made of equation(11.22) with $u(k)=0$. Equation(11.90) is a non-homogeneous linear algebraic system system of equation with $x(0)$ which is an unknown variable. Equation(11.90) has a unique solution for $x(0)$ as is required from definition11.4.2 if ,and only if there exists a finite q such that

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} = l \tag{11.91}$$

Clearly, an increase in time q improves the possibility of satisfying condition(11.91).

However, the Cayley-Hamilton theorem states that the terms CA^j, for $j \geq l$, are linearly dependent on the first l terms (i.e., on the terms C, CA, $\cdots$, CA^{l-1}). Thus, condition(11.91) holds true if and only if $q = l$, i.e., if

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{bmatrix} = l \tag{11.92}$$

Therefore, the following theorem can be achieved.

Theorem11.4.2

System described by equation(11.84b) is observable if and only if

$$rankS_o = l, \quad where \quad S_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{bmatrix} \tag{11.93}$$

Here the $l \times l$ judgment matrix S_{ou} is called the observability matrix.

In an observable system, the knowledge of the first output values $\{y(0),\quad y(1),\quad \cdots,\quad y(l-1)\}$ is sufficient to determine the initial condition $x(0)$ of the system uniquely by equation(11.90)

Example11.10 A discrete-time system can be described by equation(11.84a and 11.84b), and related matrices are as follows

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Please find the initial system state, if it exists, that can drive the system to the desired output sequence $\{y(0),\quad y(1)\} = \{1 \quad 2\}$.

Solution

Confirm the observability matrix S_o and then judge whether given system is completely observable.

$$S_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad |S_o| = 1 \neq 0$$

. Hence, given system of example11.10 is completely controllable. And the initial conditions may be determined from equation(11.90) which , for the present example, is

$$\begin{bmatrix} C \\ CA \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}$$

Namely，

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For above matrix equation, we can obtain $x_1(0)=1$ and $x_2(0)=0$.

11.4.3 Loss of controllability and observability due to sampling

As we have already known from 11.2, when sampling a continuous-time system, the corresponding discrete-time system matrices depend on the sampling period T . How does this sampling period T affect the controllability and observability of the discretized system?

For a discretized system to be controllable, it is necessary that the initial continuous-time system be controllable. This is because the control signals of the sampled-data system are only a subset of the control signals. However, the controllability may be lost for certain values of the sampling period. Hence, the initial continuous-time system may be controllable, but the equivalent discrete-time system may not. Similar problem occur for the observability of the system.

Example11.11 A continuous-time system with the harmonic oscillator can be described by the following equations:

$$\dot{x}(t)=\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}x(t)+\begin{bmatrix} 0 \\ \omega \end{bmatrix}u(t), \quad y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix}x(t)$$

Please investigate the controllability and the observability of the sampled-data (discrete-time) system which states are sampled with a sampling period T .

Solution

The discrete-time model of the harmonic oscillator is the following(see example11.6)

$$x[(k+1)T]=A(T)x(kT)+B(T)u(kT)$$
$$y(kT)=\begin{bmatrix} 1 & 0 \end{bmatrix}x(kT)$$

Replace the corresponding coefficient matrices into above equations, and then we have

$$x[(k+1)T]=\begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix}x(kT)+\begin{bmatrix} 1-\cos\omega T \\ \sin\omega T \end{bmatrix}u(kT)$$
$$y(kT)=\begin{bmatrix} 1 & 0 \end{bmatrix}x(kT)$$

Calculate the controllability matrix S_c and the observability matrix S_{ou}

$$S_c=\begin{bmatrix} B & AB \end{bmatrix}=\begin{bmatrix} 1-\cos\omega T & \cos\omega T-\cos^2\omega T+\sin^2\omega T \\ \sin\omega T & -\sin\omega T+2\sin\omega T\cos\omega T \end{bmatrix}$$

$$S_o=\begin{bmatrix} C \\ CA \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ \cos\omega T & \sin\omega T \end{bmatrix}$$

The determinants of above judgment matrices is as follows:

$$|S_c|=-2\sin\omega T(1-\cos\omega T), \quad |S_o|=\sin\omega T$$

We can notice that the controllability and observability of the discrete-time system is lost when $\omega T = n\pi$, where n is an integer, though the respective continuous-time system is both controllable and observable.

11.5 Optimal Control

For discrete-time system, plenty of problems in engineering can be solved by minimizing a measure of cost or maximizing a measure of performance. The designer has to select a suitable performance measure based on his or her understanding of the problem to include the most important performance criteria and reflect their relative importance. The designer must also select a mathematical form of the function that makes solving the optimization problem tractable. Generally, designer always solves optimizing problem by minimizing a cost function or performance measurement. In this section, aim at discrete-time system we only consider minimization for optimal methods have been given in chapter7.

11.5.1 Unconstrained and constrained optimization

1. Unconstrained optimization

We first consider the problem of minimizing a cost function or performance measure of the following form:

$$J(x) \tag{11.94}$$

Here, x is an $n \times 1$ vector of parameters to be selected. Assume that the optimal parameter vector is x^* and expand the function $J(x)$ in the vicinity of the optimum as

$$J(x) = J(x^*) + \left[\frac{\partial J}{\partial x}\right]_{x^*}^T \Delta x + \frac{1}{2!}\Delta x^T \left\{\left[\frac{\partial^2 J}{\partial x^2}\right]_{x=x^*}\right\}\Delta x + o\left(\|\Delta x\|^3\right) \tag{11.95}$$

$\Delta x = x - x^*$

Here,

$$\left[\frac{\partial J}{\partial x}\right]_{x^*}^T = \left[\frac{\partial J}{\partial x_1} \quad \frac{\partial J}{\partial x_2} \quad \cdots \quad \frac{\partial J}{\partial x_n}\right]_{x^*} \tag{11.96}$$

$$\left[\frac{\partial^2 J}{\partial x^2}\right]_{x^*} = \left[\frac{\partial^2 J}{\partial x_{ij}^2}\right]_{x^*} \tag{11.97}$$

The subscript denotes that the matrix is evaluated at optimal parameter vector x^*. At a local minimum or local maximum, the first-order terms of the expansion that appear in the gradient vector are zero.

To guarantee that the optimal point x^* is a minimum, any perturbation vector Δx away from x^* must result in an increase of the value of $J(x)$. Thus, the second term of the expansion must at least be positive or zero for x^* to have any chance of being a minimum. If the second term is positive, then we can guarantee that optimal parameter vector x^* is indeed a minimum. The sign of the second term is determined by the characteristics of the second derivative matrix. The second term is positive for any perturbation if the second derivative matrix is positive definite, and it is positive or zero if the second derivative matrix is positive semi-definite. We summarize this discussion in the following theorem.

Theorem11.5.1

If a parameter vector x^* is a local minimum of $J(x)$, then

$$\left[\frac{\partial J}{\partial x}\right]_{x^*}^{T} = 0_{1\times n} \tag{11.98}$$

A sufficient condition for x^* to be a minimum is

$$\left[\frac{\partial^2 J}{\partial x^2}\right]_{x=x^*} \geq 0 \tag{11.99}$$

Example11.12 Please obtain the least square estimation of linear resistance:

$$\upsilon = iR$$

using N noisy measurement

$$z(k) = i(k)R + \upsilon(k), \quad k = 1,2,3,\cdots,N$$

Solution

We begin to obtain the matrix equation by stacking the measurements:

$$Z = IR + V$$

in terms of the following vectors

$$Z = [z(1) \quad \cdots \quad z(N)]^T, \quad V = [\upsilon(1) \quad \cdots \quad \upsilon(N)]^T, \quad I = [i(1) \quad \cdots \quad i(N)]^T$$

We minimize the sum of the squares of the resistance errors

$$J = \sum_{k=1}^{N} e^2(k), \quad e(k) = z(k) - i(k)\hat{R}$$

Here, the caret(^) denotes the estimates. We rewrite the performance measurement in terms of vectors as

$$J = e^T e = Z^T Z - 2I^T Z \hat{R} + I^T I \hat{R}^2, \quad e = [e(1) \quad \cdots \quad e(N)]^T$$

The necessary condition for a minimum gives

$$\frac{\partial J}{\partial \hat{R}} = 2I^T Z + 2I^T I \hat{R} = 0$$

Now we have the least-squares estimate:

$$\hat{R}_{LS} = \frac{I^T Z}{I^T I}$$

The solution is indeed a minimum because the second derivative is the positive sum of the squares of the currents:

$$\frac{\partial^2 J}{\partial \hat{R}^2} = I^T I > 0$$

2.Constrained optimization

In some practical applications, the entries of the parameter vector X are subject to physical and economic constraints. Assume that the vector of parameters is to subject to the following equality constraint.

$$m(x) = 0_{m\times 1} \tag{11.100}$$

Here, we utilize Lagrange multipliers in advanced math to obtain the Lagrangian expression:

$$L(x) = J(x) + \lambda m(x) \tag{11.101}$$

Then the problem of constrained optimization is converted into the problem of unconstrained optimization. For expression(11.101), we may solve the optimal estimate in terms of equality(11.98). Here, an example is provide to demonstrate such idea.

Example11.13 A manufacture decides the production level of the two products based on maximizing profit subject to constraints on production. The manufacturer estimates profit using the simplified measure:

$$J(X) = x_1^\alpha x_2^\beta$$

Here, x_i is the quantity produced for product $i = 1,2$ and the parameters (α, β) are determined from sale data. The quantity of the two products produced cannot exceed a fixed level b. Determine the optimum production level for the two products limited by the following product constraint:

$$x_1 + x_2 = b$$

Solution

In this example, we will use above two methods to solve the optimization problem.

(1) Unconstrained optimization method

We can get x_2 expression from given constraint: $x_2 = b - x_1$

x_2 expression is substituted into $J(x)$, and then we get:

$$J(x_1) = x_1^\alpha (b - x_1)^\beta$$

The necessary condition for a minimum gives

$$\frac{\partial J}{\partial x_1} = \alpha x_1^{\alpha-1}(b - x_1)^\beta - \beta x_1^\alpha (b - x_1)^{\beta-1} = 0$$

Solve above equation and then we can get the minimum of x_1:

$$\alpha(b - x_1) - \beta x_1 = 0\text{, namely } x_1 = \frac{\alpha b}{\alpha + \beta}$$

From the production constraint we solve for production level:

$$x_2 = \frac{\beta b}{\alpha + \beta}$$

In the following, we will employ constrained optimization method of Lagrange multiplier to solve above problem.

(2) Constrained optimization method

Assume that the coefficient is λ, and then in terms of equation(11.101), we can get the following Lagrangian function.

$$L(x) = J(x) + \lambda m(x) = x_1^{\alpha} x_2^{\beta} + \lambda(x_1 + x_2 - b)$$

The necessary conditions for the minimum are

$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} + \lambda, \quad \frac{\partial L}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} + \lambda, \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 - b = 0$$

The first two conditions give

$$\alpha x_1^{\alpha-1} x_2^{\beta} = \beta x_1^{\alpha} x_2^{\beta-1}, \text{ namely } x_2 = \frac{\beta}{\alpha} x_1$$

Above equality is substituted into the third condition, and then solve

$$x_1 = \frac{\alpha b}{\alpha + \beta}, \quad x_2 = \frac{\beta b}{\alpha + \beta}$$

11.5.2 Optimal control

Some optimal control conclusions for discrete-time system have been given out in Chapter7, but the deducing course mainly focuses on continuous-time system, and the detailed narration on discrete-time system can not be provided. In this section, some simple deducing course of discrete-time system will be provided for readers to help them understand some previous conclusions.

To optimize the performance of a discrete-time dynamic system, we minimize the performance index functional:

$$J = J_f\left(x(k_f), \quad k_f\right) + \sum_{k=k_0}^{k_f-1} L\left(x(k), \quad u(k), \quad k\right) \tag{11.102}$$

Constraint condition

$$x(k+1) = Ax(k) + Bu(k), \quad k = k_0, k_1, \cdots, k_f - 1 \tag{11.103}$$

Under the condition that the system described by equation(11.103) is stable, we may change the constrained problem in equation(11.102) into unconstrained problem using Lagrange multiplier in terms of equation(11.101). And then we may get the new performance index functional:

$$J_{New} = J_f\left(x(k_f), \quad k_f\right) + \sum_{k=k_0}^{k_f-1} \left\{L\left(x(k), \quad u(k), \quad k\right) + \lambda^T(k+1)\left[Ax(k) + Bu(k) - x(k+1)\right]\right\} \tag{11.104}$$

We define the Hamilton function as

$$\begin{aligned} &H\left(x(k), \quad u(k), \quad k\right) = L\left(x(k), \quad u(k), \quad k\right) + \lambda^T(k+1)\left[Ax(k) + Bu(k)\right], \\ &k = k_0, \cdots, k_f - 1 \end{aligned} \tag{11.105}$$

And rewrite the performance index functional in terms of above Hamilton function as

$$\begin{aligned}&J_{New}=J_f\left(x\left(k_f\right),\quad k_f\right)-\lambda^T\left(k_f\right)x\left(k_f\right)+\\&\sum_{k=k_0+1}^{k_f-1}\left\{H\left(x(k),\quad u(k),\quad k\right)-\lambda^T(k)x(k)\right\}+H\left(x\left(k_0\right),\quad u\left(k_0\right),\quad k_0\right)\end{aligned} \tag{11.106}$$

Each term of the above expression can be expanded as Taylor series in the vicinity of the optimum point:

$$J_{New}=J_{New}^*+\delta_1 J_{New}+\delta_2 J_{New}+\delta_3 J_{New}+\cdots \tag{11.107}$$

Here, the sign* denotes the optimum, δ denotes a variation from the optimal, and the subscripts denote the order of the variation in the expansion(11.107). From basic calculus knowledge in the advanced math, we know that the necessary condition for a minimum or maximum is that the first-order term must be zero. For a minimum the second-order term must be positive or zero, and a sufficient condition for a minimum is a positive second term. Therefore we need to evaluate the terms of the expansion to determine necessary and sufficient conditions for a minimum. According to such idea, Hamilton function is expanded as

$$\begin{aligned}&H\left(x(k),\quad u(k),\quad k\right)=H\left(x^*(k),\quad u^*(k),\quad k\right)+\left.\frac{\partial H}{\partial x(k)}\right|_*^T\delta x(k)+\left.\frac{\partial H}{\partial u(k)}\right|_*^T\delta u(k)+\\&\left[Ax(k)+Bu(k)\right]_*^T\delta\lambda(k+1)+\delta x^T(k)\left.\frac{\partial^2 H}{\partial x^2(k)}\right|_*^T\delta x(k)+\delta u^T(k)\left.\frac{\partial^2 H}{\partial u^2(k)}\right|_*^T\delta u(k)+\\&\delta x^T(k)\left.\frac{\partial^2 H}{\partial x(k)\partial u(k)}\right|_*^T\delta u(k)+\delta u^T(k)\left.\frac{\partial^2 H}{\partial u(k)\partial x(k)}\right|_*^T\delta x(k)+\cdots\cdots\end{aligned} \tag{11.108}$$

Here δ denotes a variation from the optimal, the subscript(*) denotes the optimal value at the optimum point. The second-order terms can be written more compactly in terms of the matrix of second derivatives in the following form:

$$\begin{bmatrix}\delta x^T(k) & \delta u^T(k)\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2 H\left(x(k)\quad u(k)\quad k\right)}{\partial x^2(k)} & \dfrac{\partial^2 H\left(x(k)\quad u(k)\quad k\right)}{\partial x(k)\partial u(k)}\\ \dfrac{\partial^2 H\left(x(k)\quad u(k)\quad k\right)}{\partial u(k)\partial x(k)} & \dfrac{\partial^2 H\left(x(k)\quad u(k)\quad k\right)}{\partial u^2(k)}\end{bmatrix}\begin{bmatrix}\delta x(k)\\ \delta u(k)\end{bmatrix} \tag{11.109}$$

We expand the linear terms of the performance index functional as

$$\lambda^T(k)x(k)=\lambda^{*T}(k)x^*(k)+\delta\lambda^T(k)x^*(k)+\lambda^{*T}(k)\delta x(k) \tag{11.110}$$

The terminal polynomial can be expanded as

$$\begin{aligned}&J_f\left(x\left(k_f\right),\quad k_f\right)=J_f\left(x^*\left(k_f\right),\quad k_f\right)+\left.\frac{\partial J_f\left(x\left(k_f\right),\quad k_f\right)}{\partial x\left(k_f\right)}\right|_*^T\delta x\left(k_f\right)\\&+\delta x^T\left(k_f\right)\left[\frac{\partial^2 J_f\left(x^*\left(k_f\right),\quad k_f\right)}{\partial x^2\left(k_f\right)}\right]_*\delta x\left(k_f\right)\end{aligned} \tag{11.111}$$

Now we combine the first-order terms to obtain the increment:

$$\delta_1 J_{New} = \sum_{k=k_0+1}^{N-1}\left\{\left[\left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial x(k)}\right|_* - \lambda^*(k)\right]^T \delta x(k) + \left[Ax^*(k)+Bu^*(k)-x^*(k+1)\right]^T \delta\lambda(k+1)\right\} +$$
$$\left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial u(k)}\right|_*^T \delta u(k) + \left[\frac{\partial J_f(x(k_f),\quad k_f)}{\partial x(k_f)} - \lambda(k_f)\right]_*^T \delta x(k_f)$$

(11.112)

From equation(11.107), we know that a necessary condition for a minimum of the performance index functional is that the increment must be equal to zero for any combination of variations in its arguments. A zero for any combination of variations occurs only if the coefficient of each increment is equal to zero. Equating each coefficient to zero, we have the following necessary conditions:

$$x^*(k+1) = Ax^*(k) + Bu^*(k), \quad k = k_0, \cdots, k_f - 1 \tag{11.113}$$

$$\lambda^*(k) = \left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial x(k)}\right|_*, \quad k = k_0, \cdots, k_f - 1 \tag{11.114}$$

$$\left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial u(k)}\right|_* = 0, \quad k = k_0, \cdots, k_f - 1 \tag{11.115}$$

$$\left[\frac{\partial J_f(x(k_f),\quad k_f)}{\partial x(k_f)} - \lambda(k_f)\right]_*^T \delta x(k_f) = 0 \tag{11.116}$$

If the terminal point is fixed, then its perturbation is zero and the terminal optimality condition is satisfied. Otherwise, we have the terminal conditions

$$\lambda(k_f) = \frac{\partial J_f(x(k_f),\quad k_f)}{\partial x(k_f)} \tag{11.117}$$

Table11.5 Optimality conditions

Condition	Equation
State equation	$x^*(k+1) = Ax^*(k) + Bu^*(k), \quad k = k_0, \cdots, k_f - 1$
Costate equation	$\lambda^*(k) = \left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial x(k)}\right\|_*, \quad k = k_0, \cdots, k_f - 1$
Minimum principle	$\left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial u(k)}\right\|_* = 0, \quad k = k_0, \cdots, k_f - 1$

Similarly, conditions can be developed for the case of a free initial state with an initial cost added to the performance index functional.

The necessary conditions for a minimum of equation(11.113) through (11.115) are known, respectively, as the **state equation**, the **costate equation**, and the **pontryagin minimum principle**. The minimum principle tells us that the cost is minimized by choosing the control that minimize the Hamiltonian. This condition can be generalized to problems where the control is constrained, in which case the solution can be at the boundary of the allowable control region. We summarize the necessary conditions for an optimum in table11.5.

The second term is

$$\delta_2 J_{New} = \sum_{k=k_0}^{k_f-1} \begin{bmatrix} \delta x^T(k) & \delta u^T(k) \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial x^2(k)} & \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial x(k)\partial u(k)} \\ \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial u(k)\partial x(k)} & \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial u^2(k)} \end{bmatrix} \begin{bmatrix} \delta x(k) \\ \delta u(k) \end{bmatrix}$$

$$+ \delta x^T(k_f) \left[\frac{\partial^2 J_f(x(k_f),\quad k_f)}{\partial x^2(k_f)} \right]_* \delta x(k_f)$$

(11.117)

For the second-order term to be positive or at least zero for any perturbation, we need the matrix to be positive definite, or at least positive semidefinite. We have the sufficient condition:

$$\begin{bmatrix} \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial x^2(k)} & \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial x(k)\partial u(k)} \\ \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial u(k)\partial x(k)} & \dfrac{\partial^2 H(x(k),\quad u(k),\quad k)}{\partial u^2(k)} \end{bmatrix}_* > 0 \tag{11.118}$$

If the matrix in equation(11.118) is positive semidefinite, then the condition is necessary for a minimum but not sufficient because there will be perturbation for which the second-order terms are zero. Higher-order terms are then needed to determine if these perturbations result in an increase in the performance index.

For a fixed terminal state, the corresponding perturbations are zero and no additional conditions are required. For a free terminal state, we have the additional condition

$$\frac{\partial^2 J(x(k_f),\quad k_f)}{\partial x^2(k_f)} > 0 \tag{11.119}$$

Example11.14 We consider the following scalar system:

$$x(k+1) = ax(k) + bu(k), \quad x(0) = x_0, \quad b > 0$$

If the system is required to reach the zero state, find the control that minimizes the control effort to reach the desired final state.

Solution

For minimum control effort, we have the performance index functional as follows:

$$J = \frac{1}{2}\sum_{k=0}^{k_f} u^2(k)$$

Hamilton function may be constructed by

$$H(x(k),\quad u(k),\quad k) = \frac{1}{2}u^2(k) + \lambda(k+1)[ax(k) + bu(k)], \quad k = 0,1,\cdots,k_f - 1$$

For minimum time, we minimize the Hamilton by selecting the control

$$u^*(k) = -b\lambda^*(k+1)$$

The costate equation is

$$\lambda^*(k) = \left.\frac{\partial H(x(k),\quad u(k),\quad k)}{\partial x(k)}\right|_* = a\lambda^*(k+1), \quad k = 0,\cdots,k_f - 1$$

And its solution can be confirmed as

$$\lambda^*(k) = a^{k_f - k}\lambda^*(k_f), \quad k = 0,\cdots,k_f - 1$$

Hence, the optimal control input can be written as

$$u^*(k) = -b\lambda^*(k+1) = -ba^{k_f-k-1}\lambda^*(k_f), \quad k = 0,\cdots,k_f - 1$$

Next, we consider the iteration of state equation with optimal control.

$$x^*(1) = ax(0) + bu^*(0) = ax_0 - b^2 a^{k_f-1}\lambda^*(k_f)$$
$$x^*(2) = ax(1) + bu^*(1) = a^2 x_0 - b^2\left(a^{k_f} + a^{k_f-2}\right)\lambda^*(k_f)$$
$$x^*(3) = ax(2) + bu^*(2) = a^3 x_0 - b^2\left(a^{k_f+1} + a^{k_f-1} + a^{k_f-3}\right)\lambda^*(k_f)$$
$$\vdots$$
$$x^*(k_f) = ax(k_f - 1) + bu^*(k_f - 1) = a^{k_f} x_0 - b^2\left(\left(a^2\right)^{k_f-1} + \cdots + a^2 + 1\right)\lambda^*(k_f) = 0$$

Solve the last equation for the terminal Lagrange multiplier:

$$\lambda^*(k_f) = \frac{\left(a^{k_f}/b^2\right)x_0}{\left(a^2\right)^{k_f-1} + \cdots + a^2 + 1}$$

Substituting for the terminal Lagrange multiplier gets the optimal control

$$u^*(k) = -ba^{k_f-k-1}\lambda^*(k_f) = -\frac{\left(a^{2k_f-k-1}/b\right)x_0}{\left(a^2\right)^{k_f-1} + \cdots + a^2 + 1}, \quad k = 0,1,2,\cdots,k_f - 1$$

For an open-loop stable system, we can write the control in the following form:

$$u^*(k) = -\frac{\left(a^2 - 1\right)\left(a^{2k_f-k-1}/b\right)x_0}{\left(a^2\right)^{k_f} - 1}, \quad k = 0,1,2,\cdots,k_f - 1; |a| < 1$$

Note that the closed-loop system dynamics of the optimal system are time varying even though the open-loop system is time invariant. The closed-loop system is given by

$$x(k+1) = ax(k) - \frac{\left(a^{2k_f-k-1}/b\right)x_0}{\left(a^2\right)^{k_f-1} + \cdots + a^2 + 1}, \quad k = 0,1,2,\cdots,k_f - 1; x(0) = x_0, b > 0$$

11.5.3 Linear quadratic regulator

The choice of performance index in optimal control determines the performance of the system and the complexity of the optimal control problem. The most popular choice for the performance index is a quadratic function of the state variable and the control inputs. The performance index functional have the following form:

$$J = \frac{1}{2}x^T(k_f)P(k_f)x(k_f) + \frac{1}{2}\sum_{k=k_0}^{k_f-1}\left(x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\right) \qquad (11.120)$$

Here the matrices $P(k_f)$, $Q(k)$, $k = k_0, k_1, \cdots, k_f - 1$ are positive definite symmetric $n \times n$ matrices, and the matrix $R(k)$, $k = k_0, k_1, \cdots, k_f - 1$ are positive definite symmetric $m \times m$ matrix. This choice is known as the ***linear quadratic regulator***, and in addition to its desired mathematical properties it can be physically justified. In a regulator problem, the purpose is to maintain the system close to the zero state and the quadratic functional of the state variable is a measurement error. On the other hand, the effort needed to minimize that the error must be also minimized, as must the control effort represented by a quadratic functional of the controls. Thus the quadratic performance index measurement achieves a compromise between minimizing the regulator error and minimizing the control effort.

To obtain the necessary and sufficient conditions for the linear quadratic regulator, we may use the results of section11.5.2. In terms of equation(11.105), we first write an Hamilton function of the following form:

$$H(x(k),\quad u(k),\quad k) = \frac{1}{2}x^T(k)Q(k)x(k) + \frac{1}{2}u^T(k)R(k)u(k) + \lambda^T(k+1)[Ax(k) + Bu(k)],$$
$$k = k_0, k_1, \cdots, k_f - 1 \tag{11.121}$$

Equation(11.121) can be also found in section7.9. Related conclusions in section7.9 are listed out as follows:

Costate equation:

$$\lambda^*(k) = Q(k)x^*(k) + A^T\lambda^*(k+1), \qquad k = k_0, k_1, \cdots, k_f - 1 \tag{11.122}$$

If control input is not constrained, the control equation is the following:

$$R(k)u^*(k) + B^T\lambda^*(k+1) = 0 \tag{11.123}$$

Hence, the optimum control expression may be acquired:

$$u^*(k) = -R^{-1}(k)B^T\lambda^*(k+1) \tag{11.124}$$

After equation(11.124) is substituted into state equation, we can get the optimal dynamics:

$$x^*(k+1) = Ax^*(k) - B^*R^{-1}(k)B^T\lambda^*(k+1), \quad k = k_0, k_1, \cdots, k_f - 1 \tag{11.125}$$

From equation(11.118), the following condition can be given:

$$\begin{bmatrix} Q(k) & 0_{n\times m} \\ 0_{m\times n} & R(k) \end{bmatrix}_* > 0 \tag{11.126}$$

A sufficient condition for a minimum is that Q must be positive definite because R is positive definite. A necessary condition is that Q must be positive semidefinite.

Finite time state regulator and infinite time state regulator have been narrated for a discrete-time system. Here these content will not be given again, and readers may consult representation in section7.9.

11.5.4 Steady-state quadratic regulator

Implementing the linear quadratic regulator in practice is rather complicated because people need to calculate and store gain matrices. In some applications, it is possible to simplify implementation considerably using the steady-state gain exclusively in place of the optimal gains.This solution is only optimal if the summation interval in the performance index measurement, known as the planning horizon, is infinite. For a finite planning horizon, the simplified solution is suboptimal (i.e., require a higher value of the performance measurement) but often performs almost as well as the optimal control. Thus, it is possible to retain the performance of optimal control without the burden of implementing it if we solve the steady-state regulator problem with the performance measurement of the following form.

$$J=\frac{1}{2}\sum_{k=k_0}^{\infty}\left[x^T(k)Qx(k)+u^T(k)Ru(k)\right] \tag{11.127}$$

Here, we assume that the weighting matrices Q and R are constant with Q positive semidefinite and R positive definite. And therefore we can decompose the matrix Q as

$$Q=Q_a^TQ_a \tag{11.128}$$

Here, Q_a is an square root matrix. In terms of an equivalent measurement vector, we may write out the state error terms of measurements.

$$y(k)=Q_ax(k) \tag{11.129a}$$

$$x^T(k)Qx(k)=y^T(k)y(k) \tag{11.129b}$$

The matrix Q_a is positive definite for Q positive definite and positive semidefinite for Q positive semidefinite. In the latter case, large state vectors can be mapped by Q_a to zero $y(k)$, and a small performance index measurement can not guarantee small errors or even stability. We regard the vector $y(k)$ as a measurement of the state and recall the observability condition from chapter5.

If the matrices A and Q_a are detectable, then state vector $x(k)$ decays asymptotically to zero with output $y(k)$. Hence, the detectability condition is required for acceptable steady-state regulator behavior. If the matrices A and Q_a is observable, then it is also detectable and the system behavior is acceptable. For example, if the matrix Q_a is positive definite, there is a one-to-one mapping between the states X and the system output Y; and the matrices A and Q_a are always observable. Hence, a positive definite matrix Q (and Q_a) is sufficient for acceptable system behavior.

If the observability condition is satisfied and the system is stable, then the Riccati equation does not diverge and a steady-state condition is reached. The resulting algebraic

Riccati equation is gained in the following form:

$$P = A^T\left[P - PB\left(B^T PB + R\right)^{-1} B^T P\right]A + Q \tag{11.130}$$

It can be shown that, under these conditions, the algebraic Riccati equation has a unique positive definite solution. However, the equation is clearly difficult to be solved in general and is typically solved numerically.

The optimal state feedback corresponding to the steady-state regulator is

$$u^*(k) = -Kx^*(k) \tag{11.131a}$$

$$K = \left(R + B^T PB\right)^{-1} B^T PA \tag{11.131b}$$

1.Output quadratic regulator

In most practical applications, the designer is interested in optimally controlling the system output $y(k)$ rather than the state $x(k)$. To optimally control the output, we need to consider a performance index of the following form.

$$J = \frac{1}{2}\sum_{k=k_0}^{\infty}\left[y^T(k)Q_y y(k) + u^T(k)Ru(k)\right] \tag{11.132}$$

From equation(11.129), we observe that this is equivalent to the original performance measure of equation(11.127) with the state weight matrix

$$Q = C^T Q_y C = C^T Q_{ya}^T Q_{ya} C = Q_a^T Q_a \tag{11.133a}$$

$$Q_a = Q_{ya} C \tag{11.133b}$$

Here, Q_{ya} is the square root of the output weight matrix. As in the state quadratic regulator, the Riccati equation for the output regulator can be solved using the MATLAB commands. For a stabilizability system $(A,\ \ B)$, the solution exists provided that the matrices$(A,\ \ Q_a)$ is detectable with Q_a as in equation(11.133).

2.Linear quadratic tracking controller

The minimization of the performance index in equation(11.127) yields an optimal state feedback matrix K that minimizes the integral square error and control effort in equation(11.129). The error is defined relative to the zero state. In many practical applications, the system is required to follow a specified function of time. The design of a controller to achieve that this objective is known as the ***tracking problem***. If the control task is to track a nonzero constant reference input, we can exploit the techniques to solve the new optimal regulator problem.

The tracking or servo problem can be solved using an additional gain matrix for the reference input, thus providing an additional degree of freedom in our design. For a square system(equal number of inputs and outputs), we can implement the degree of freedom scheme with the reference gain matrix.

$$F = \left[C\left(E_{unit} - A_{cl}\right)^{-1} B\right]^{-1} \tag{11.134}$$

Here, $A_{cl} = A - BK$

All other equations for the linear quadratic regulator are not changed.

Alternatively, to improve the robustness, we may introduce integral control but at the expense of a high-order controller. In particular, we consider the following state-space equations with the integral control gain $\overline{K}$.

$$\begin{gathered} x(k+1)=Ax(k)+Bu(k) \\ \overline{x}(k+1)=\overline{x}(k)+r(k)-y(k) \\ y(k)=Cx(k) \\ u(k)=-Kx(k)-\overline{K}\,\overline{x}(k) \end{gathered} \tag{11.135}$$

This yields the closed-loop state space system

$$\tilde{x}(k+1)=\left(\tilde{A}-\tilde{B}\tilde{K}\right)\tilde{x}(k)+\begin{bmatrix} 0 \\ E_l \end{bmatrix} r(k) \tag{11.136a}$$

$$y(k)=\begin{bmatrix} C & 0 \end{bmatrix}\tilde{x}(k) \tag{11.136b}$$

Here, $\tilde{x}(k)=\begin{bmatrix} x(k) & \overline{x}(k) \end{bmatrix}^T$. Matrices are given by

$$\tilde{A}=\begin{bmatrix} A & 0 \\ -C & E_l \end{bmatrix},\quad \tilde{B}=\begin{bmatrix} B \\ 0 \end{bmatrix},\quad \tilde{C}=\begin{bmatrix} C & 0 \end{bmatrix}, \tilde{K}=\begin{bmatrix} K & \overline{K} \end{bmatrix} \tag{11.137}$$

The state feedback gain matrix $\tilde{K}$ can be computed as

$$\tilde{K}=\left(R+\tilde{B}^T P\tilde{B}\right)^{-1}\tilde{B}^T P\tilde{A} \tag{11.138}$$

11.6 Discrete-time Controller Design

The classical discrete-time controller design methods are categorized as indirect and direct techniques. For indirect techniques, a discrete-time controller is determined indirectly as follows. Initially, the continuous-time controller is designed in the s -domain, using well-known classical techniques(i.e., root locus, Bode, Nyquist, etc). Then, based on the continuous-time controller, the discrete-time controller may be calculated using one of discretization techniques presented in chapter2. For direct techniques, firstly, a discrete-time mathematical model of the continuous-time system under control is confirmed. Subsequently, the design is carried out in the z domain wherein the discrete-time controller is directly determined. The design in the z domain may be done either using the root locus or the Bode and Nyquist diagrams.

In nature, the design methods employed by controllers is mainly utilized to confirm the mathematical expression of controller $G_c(z)$ or $G_c(s)$.

11.6.1 Time response specifications of continuous-time system

The time response specifications of continuous-time system has been given and explained in detail in classical control theory. Here, we only simply list out the related conclusions from classical control theory; and main specifications on controller have overshoot, rising time and settling time referring to second-order system.

1.Overshoot

One of the basic characteristic of the transient response of a system is the overshoot σ, which depends mainly on the damping factor ς. In the case of a second-order system, without zeros, i.e., for a system with a transfer function of the following form:

$$H(s)=\frac{\omega_n}{s^2+2\varsigma\omega_n s+\omega_n^2} \tag{11.139}$$

Here, ω_n is the nature frequency of a system, ζ is the damping ratio of system. The overshoot percentage is

$$\sigma\%=\frac{c(t_p)-c(\infty)}{c(\infty)}\times 100\%=\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\times 100\% \tag{11.140a}$$

Peak time t_p is defined as the time required for the response of the system to peak at first time. $t_p=\frac{\pi}{\omega_d}=\frac{\pi}{\sqrt{1-\zeta^2}\omega_n}$. Generally, when $\zeta=0.4\sim 0.8$, the corresponding overshoot range can be confirmed: $\sigma\%=25\%\sim 2.5\%$.

2.Rising time

Rising time T_r is defined as the time required for the response of the system to rise from 0.1 to 0.9 of its final value. For all values of ς around 0.5, the rise time is approximately given by

$$T_r\approx 1.8/\omega_n \tag{11.140b}$$

Hence, satisfying the above relation for the rising time, the nature frequency ω_n should satisfy the condition

$$\omega_n\ge 1.8/T_r \tag{11.140c}$$

3.Settling time

Settling time T_s is defined as the time required for the response to remain close to the final value. The settling time T_s is given by the relation:

$$T_s=\Delta/\varsigma\omega_n \tag{11.140d}$$

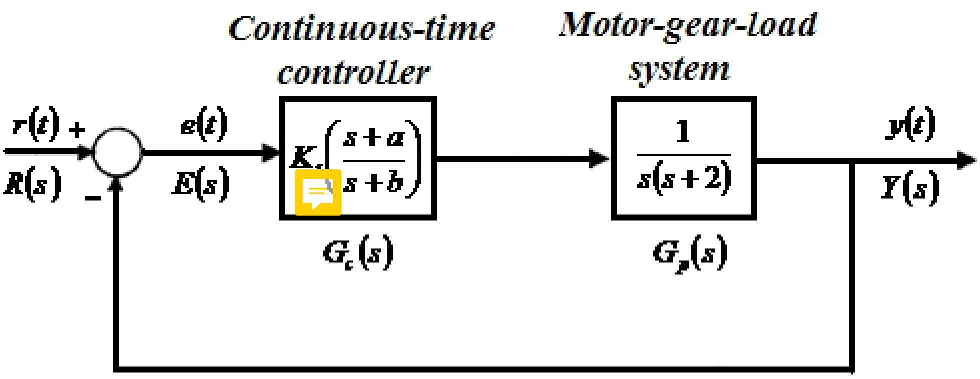
Here, Δ is a constant. In the case of an error tolerance about 1%, the constant Δ is 4.6, whereas in the case of an error tolerance about 2%, the constant Δ takes on the value4. Hence, if we desire that the settling time be smaller than a specified value and for an error tolerance about 1%, then

$$\varsigma\omega_n \geq 4.6/T_s \qquad (11.140e)$$

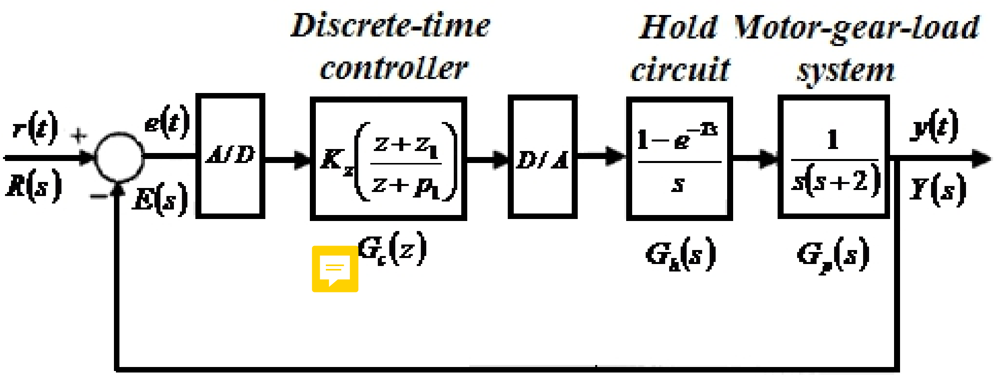
Note that in terms of Shannon law, the sampling frequency f must be at least twice the highest frequency of the frequency spectrum of the continuous-time input signal. In practice, for a wide class of systems, the selection of the sampling period $T = 1/f$ is made using the following approximate method: assume that q is the smallest time constant of the system; then, sampling period T is chosen such that $T \in [0.1q, \quad 0.5q]$.

Example11.15 The transfer function of a motor-gear-load system is the following:

$$G_p = \frac{1}{s(s+2)}$$



(a)Continuous-time closed-loop system



(a)Discrete-time closed-loop system

Figure11.12 Block diagram of motor-gear-load closed-loop system of position control servomechanism

To realize the servo control, a continuous controller is introduced which satisfies the design requirements and which has the form:

$$G_c(s) = K_s\left[\frac{s+a}{s+b}\right] = 101\left[\frac{s+2}{s+6.7}\right]$$

Please determine that a discrete-time controller $G_c(z)$ of the closed-loop system needs to satisfy the design requirement, which sampling time is 2Sec. The block diagram is shown in figure11.12.

Solution

The transfer function $H(s)$ of the closed-loop continuous-time system in figure11.12(a) is

$$H(s)=\frac{Y(s)}{R(s)}=\frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}=\frac{101(s+2)}{s(s+2)(s+6.7)+101(s+2)}=\frac{101s+202}{s^3+8.7s^2+114.4s+202}$$

To find $G_c(z)$ of the closed-loop system in figure11.12(b), it is sufficient to discretize $G_c(s)$ in example11.15. Here, we employ pole-zero matching method given in section11.2.3 of Chapter11. According to this method, the controller $G_c(z)$ has the following form:

$$G_c(z)=K_z\left(\frac{z+z_1}{z+p_1}\right)$$

Here, the pole $s=-6.7$ of $G_c(s)$ is mapped into the pole $z=-p_1$ of $G_c(z)$ and the zero $s=-2$ of $G_c(s)$ is mapped into the zero $z=-z_1$ of $G_c(z)$. Then we have that

$$p_1=-e^{-bT}=-e^{-6.7\times 0.2}=-e^{-1.34}=-0.264$$

$$z_1=-e^{-aT}=-e^{-2\times 0.2}=-e^{-0.4}=-0.67$$

The constant K_z of $G_c(z)$ is calculated so that the zero frequency amplification constants of $G_c(z)=G_c(s)G_h(s)$ are the same, which relation can be seen in equation(11.55).

$G_c(z)\big|_{z=1}=G_c(s)G_h(s)\big|_{s=0}$, namely, $K_z\left(\frac{1-0.67}{1-0.264}\right)=101\left(\frac{0+2}{0+6.7}\right)\left(\frac{2}{0+10}\right)$

Hence, $K_z=13.6$. It is noted that in the relation above the value of $G_h(s)$ for $s=0$ was taken into consideration to obtain the total zero frequency amplification for the continuous-time controller. Hence

$$G_c(z)=13.6\left(\frac{z-0.67}{z-0.264}\right)$$

For figure11.12(a), the discrete-time transfer function of $\hat{G}(s)=G_h(s)G_p(s)$ is determined for $T=0.2$.

$$\hat{G}(z)=Z\{G_h(s)G_p(s)\}=Z\left\{\left[\frac{1-e^{-0.2s}}{s}\right]\left[\frac{1}{s(s+2)}\right]\right\}=(1-z^{-1})Z\left\{\left[\frac{1}{s^2(s+2)}\right]\right\}$$

$$=(1-z^{-1})Z\left\{\frac{0.5}{s^2}-\frac{0.25}{s}+\frac{0.25}{s+2}\right\}=\frac{z-1}{z}\left[\frac{0.1}{(z-1)^2}-\frac{0.25z}{z-1}+\frac{0.25z}{z-e^{-0.4}}\right]$$

$$=\frac{0.0176(z+0.876)}{(z-1)(z-0.67)}$$

The transfer function of the closed-loop system in figure11.12(b) would be

$$H(z)=\frac{G_c(z)\hat{G}(z)}{1+G_c(z)\hat{G}(z)}=\frac{0.239z^{-1}\left(1+0.876z^{-1}\right)}{\left(1-0.264z^{-1}\right)\left(1-z^{-1}\right)+0.239z^{-1}\left(1+0.876z^{-1}\right)}$$
$$=\frac{0.239z^{-1}+0.209z^{-2}}{1-1.025z^{-1}+0.437z^{-2}}$$

If a computer is introduced into closed-loop discrete-time system, the computer can be considered as a discrete controller, which may substitute the controller in figure11.12(b). We assume that the transfer function of computer is represented by

$$D(z)=\frac{U(z)}{E(z)} \tag{11.141}$$

How do we confirm the transfer function $D(z)$ from given transfer function $G_c(z)$?

Two methods may be employed here: (1) the $G_c(s)-to-D(z)$ conversion method; (2) the root locus z plane method.

One method for determining $D(z)$ firstly determines a discrete-time controller $G_c(s)$ for a given process $G_p(s)$ for a given system. Then, the controller is converted to $D(z)$ for the given sampling period T. This design method is called the $G_c(s)-to-D(z)$ conversion method.

We consider the first-order controller

$$G_c=K\frac{s+a}{s+b} \tag{11.142a}$$

and assume digital controller

$$D(z)=C\frac{z+A}{z+B} \tag{11.142b}$$

We determine the z-transform corresponding to $G_c(s)$ and set it equal to $D(z)$ as

$$Z\{G_c(s)\}=D(z) \tag{11.143a}$$

In terms of pole-zero matching method(see section11.2), the relation between these two transfer functions is $A=-e^{-aT}$, $B=-e^{-bT}$.

When $s=0$, from equation(11.55) we have

$$C\frac{1+A}{1+B}=K\frac{a}{b} \tag{11.144}$$

11.6.2 Controller design using the frequency response method

1. Introduction

The well-established frequency domain design techniques for continuous-time system have been narrated in classical control theory. Such techniques may be extended to cove the case of the discrete-time systems. At first, we might think of carrying out this extension by using the relation $z = e^{sT}$. Utilizing this relation, the simple and easy-to-use logarithmic curves of the Bode diagrams for the continuous-time case cease to hold for the discrete-time systems. To maintain the simplicity of the logarithmic curves for the discrete-time systems, we can employ the following bilinear transformation:

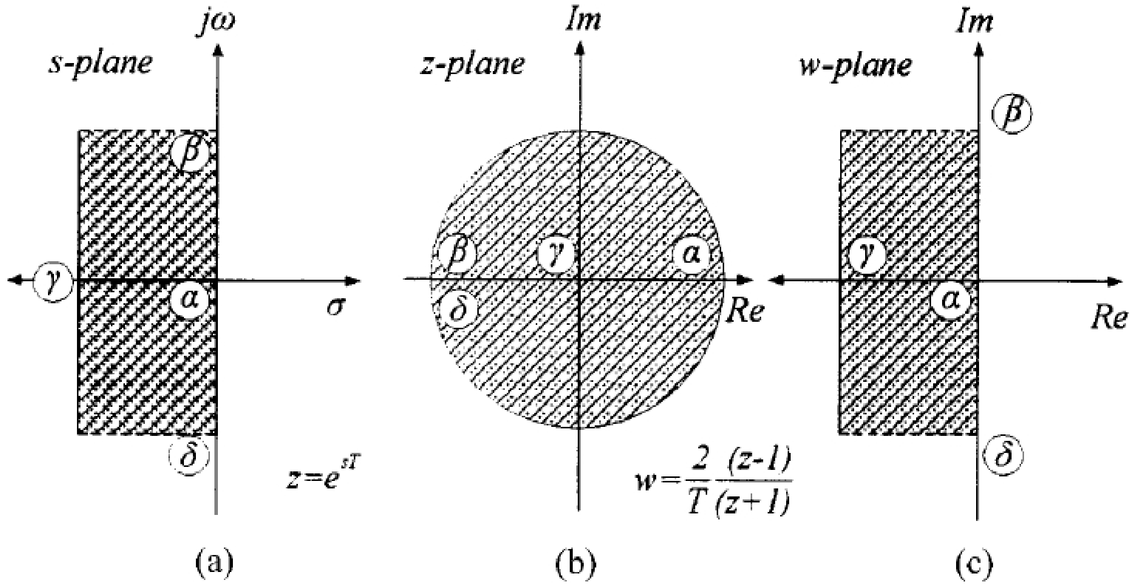


Figure11.13 Mapping form the s-plane and from z-plane to the w-plane

$$z = \frac{1 + Tw/2}{1 - Tw/2} \quad \text{or} \quad w = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \tag{11.145}$$

The transformation of a function in s domain to a function in z domain based on the relation $z = e^{sT}$ and , subsequently, the transformation of the resulting function in z domain to a function of w based on the relation(11.145), are presented in figure11.13. The figure11.13 shows that the transformation of the left-half complex plane on the s-plane transformation into the unit circle in the z-plane via the relation $z = e^{sT}$, whereas the unit circle on the z-plane transformation into the left-half complex plane in the w-plane, via the bilinear transformation(11.145).

At first sight, it seems that the frequency response would be the same in both the s and w domain. This is actually true, with the only difference that the scales of the frequencies w and υ are distorted, where υ is the (hypothetical or abstract) frequency in the w domain. This frequency "distortion" may be observed if in equation(11.145) we set $w = j\upsilon$ and $z = e^{j\omega T}$, yielding

$$w\Big|_{w=j\upsilon} = j\upsilon = \frac{2}{T}\left(\frac{z-1}{z+1}\right)\Bigg|_{z=e^{j\omega T}} = \frac{2}{T}\left(\frac{e^{j\omega T}-1}{e^{j\omega T}+1}\right) = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

Therefore

$$\upsilon = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right) \tag{11.146}$$

In Taylor series, above equation is developed into the following form:

$$\tan\left(\frac{\omega T}{2}\right) = \frac{\omega T}{2} - \frac{(\omega T)^3}{8} + \cdots$$

For small value of ωT we have that $\tan(\omega T/2) \approx \omega T/2$. Replacing this result into equation(11.146), we have

$$\upsilon = \omega \quad \text{for small } \omega T \tag{11.147}$$

Therefore, the frequencies ω and υ are linearly related if the product ωT is small. For greater ωT , equation(11.147) does not hold true. Note that the frequency range $-\omega_s/2 \le \omega \le \omega_s/2$ in the s domain corresponds to the frequency range $-\infty \le \upsilon \le \infty$ in the w domain, where ω_s is defined by the relation $(\omega_s/2)(T/2) = \pi/2$.

The graphics of equation(11.146) is shown in figure11.14.

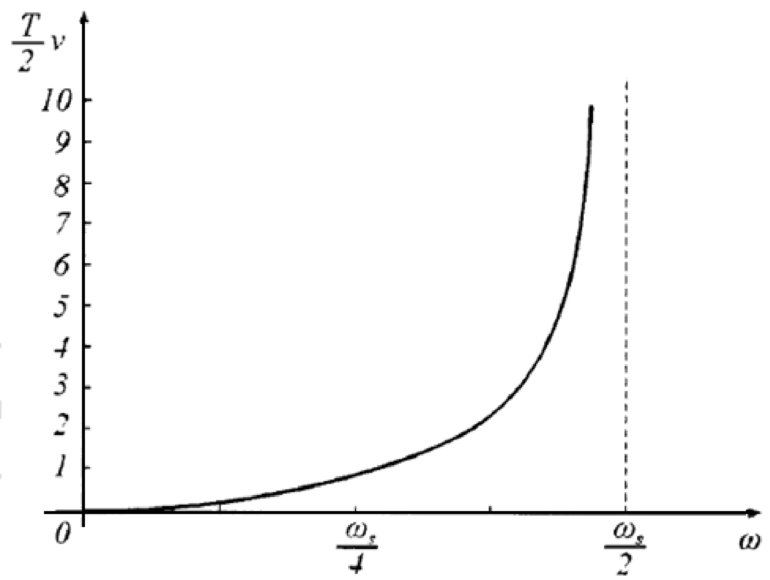


Figure11.14 Graphical representation of equation(11.146)

2.Bode diagram

Using the above results, we may readily design the discrete-time controllers using Bode diagrams. To this end, consider the closed-loop system shown in figure11.15.

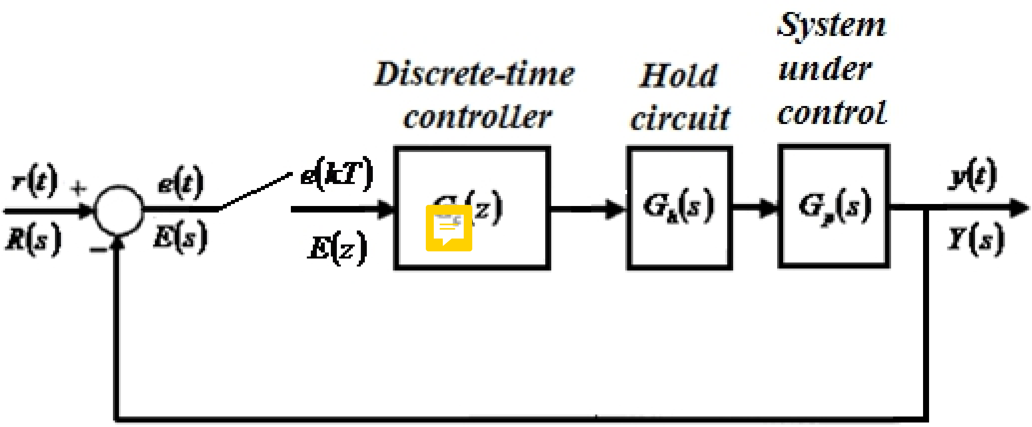


Figure11.15 Closed-loop discrete-time system

Then, the following five basic steps for designing a discrete-time controller are as follows:

Step1. Determine $\hat{G}(z)$ from the relation $\hat{G}(z)=Z\left\{\hat{G}(s)\right\}=Z\{G_h(s)G_p(s)\}$;

Step2. Determine $\hat{G}(w)$ using the bilinear transformation(11.145), yielding

$$\hat{G}(w)=\hat{G}(z)\Big|_{z=(1+Tw/2)/(1-Tw/2)} \tag{11.148a}$$

Step3. Set $w=j\upsilon$ in $\hat{G}(w)$ and draw the Bode diagrams of $\hat{G}(j\upsilon)$, for small ω, we can draw the Bode diagram of $\hat{G}(j\omega)$;

Step4. Determine the controller $G_c(w)$ using similar techniques to those applied for continuous-time system.

Step5. Determine $G_c(z)$ from $G_c(w)$ using the bilinear transformation(11.145), yielding

$$G_c(z)=G_c(w)\big|_{w=(2/T)[(z-1)/(z+1)]} \tag{11.148b}$$

Note that the specifications for the bandwidth are transformed from the s domain to the w domain using relation(11.146). Thus, if ω_b is the desired frequency bandwidth, then the design in the w domain must be carried out for a frequency bandwidth υ_b, where

$$\upsilon_b=\frac{2}{T}\tan\left(\frac{\omega_b T}{2}\right) \tag{11.148c}$$

Example11.16 Consider the position servomechanism shown in figure11.12(b) of example11.15.

Please design a controller $G_c(z)$ such that the closed-loop system satisfies the following specifications:gain margin: $K_g \geq 25dB$, phase margin: $\varphi_p \geq 70^o$, and velocity error constant $K_v = 1\sec^{-1}$. The sampling period T is chosen to be 0.1sec.

Solution

Step1. Determine $\hat{G}(z)$ from the relation $\hat{G}(z) = Z\left\{\hat{G}(s)\right\} = Z\{G_h(s)G_p(s)\}$;

$$\hat{G}(z) = Z\left\{\hat{G}(s)\right\} = Z\{G_h(s)G_p(s)\} = Z\left\{\frac{1-e^{-Ts}}{s}\frac{1}{s(s+2)}\right\} = \left(1-z^{-1}\right)Z\left\{\frac{1}{s^2(s+2)}\right\}$$

$$= 0.0047z^{-1}\left[\frac{1+0.935z^{-1}}{\left(1-z^{-1}\right)\left(1-0.819z^{-1}\right)}\right] = 0.0047\frac{z+0.935}{(z-1)(z-0.819)}$$

Step2. Determine $\hat{G}(w)$.

For $T = 0.1\sec$, the bilinear transformation(11.145) becomes

$$z = \frac{1+(Tw/2)}{1-(Tw/2)} = \frac{1+0.05w}{1-0.05w}$$

Replacing above bilinear transformation into $\hat{G}(z)$, we have the following form

$$\hat{G}(w) = \frac{0.0047\left(\dfrac{1+0.05w}{1-0.05w}+0.935\right)}{\left(\dfrac{1+0.05w}{1-0.05w}-1\right)\left(\dfrac{1+0.05w}{1-0.05w}-0.8187\right)} = \frac{0.5(1+0.00167w)(1-0.05w)}{w(1+0.5w)}$$

Step3. Draw the gain and phase Bode diagrams of $\hat{G}(j\upsilon)$ shown in figure11.16.

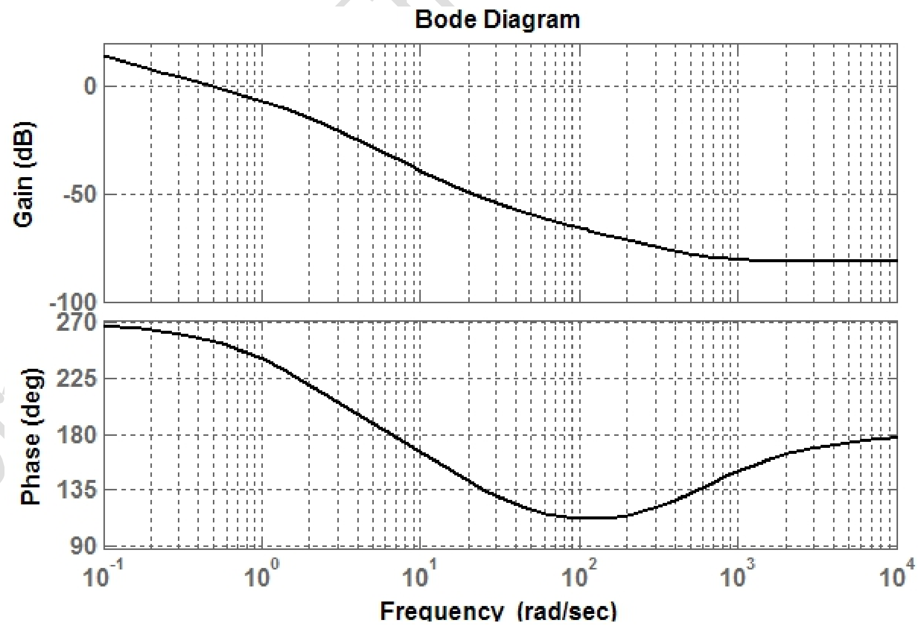


Figure11.16 Gain and phase margins of $\hat{G}(w)$

Step4. Determine the controller $G_c(w)$.

We choose the following for the controller $G_c(w)$ where a and b are constants.

$$G_c(w)=K\left(\frac{1+aw}{1+bw}\right)$$

Then we can write out the open-loop transfer function

$$G_c(w)\hat{G}(w)=K\left(\frac{1+aw}{1+bw}\right)\left[\frac{0.5(1+0.00167w)(1-0.05w)}{w(1+0.5w)}\right]$$

From the definition of the velocity error constants K_v, we have

$$K_v=\lim_{w\to 0}\left[wG_c(w)\hat{G}(w)\right]=0.5K=1$$

Therefore, $K=2$. The parameters a and b can be determined by applying the respective(equivalent) techniques of continuous-time systems, which yield $a=0.8$ and $b=0.5$. Hence

$$G_c(w)=2\left(\frac{1+0.8w}{1+0.5w}\right)$$

The open-loop transfer function is

$$G_c(w)\hat{G}(w)=2\left(\frac{1+0.8w}{1+0.5w}\right)\left[\frac{0.5(1+0.00167w)(1-0.05w)}{w(1+0.5w)}\right]$$

Step5. Determine $G_c(z)$ from $G_c(w)$.

From bilinear transformation for $T=0.1\sec$, we can get

$$w=\frac{2}{T}\left(\frac{z-1}{z+1}\right)=\frac{2}{0.1}\left(\frac{z-1}{z+1}\right)=20\left(\frac{z-1}{z+1}\right)$$

Thus, $G_c(z)$ has the following form:

$$G_c(z)=2\left(\frac{1+0.8w}{1+0.5w}\right)=2\left[\frac{1+0.8\times 20\times\left(\frac{z-1}{z+1}\right)}{1+0.5\times 20\times\left(\frac{z-1}{z+1}\right)}\right]=3.09\left(\frac{z-0.882}{z-0.818}\right)=3.09\left(\frac{1-0.882z^{-1}}{1-0.818z^{-1}}\right)$$

3. Nyquist diagrams

Consider a closed-loop system with an open-loop transfer function $\tilde{G}(z)=Z\{G(s)F(s)\}$. Since the $z-$ and $s-$ domains are related via the relation $z=e^{sT}$, it follows that the Nyquist diagram of $\tilde{G}(z)$ would be the diagram of $\tilde{G}(e^{sT})$, as s traces the Nyquist path. In the $z-$ domain, the Nyquist path is given by the relation:

$$z=e^{sT}\Big|_{s=j\omega}=e^{j\omega T} \tag{11.149}$$

Therefore, the Nyquist pathe in the z domain is the unit circle. To apply the Nyquist stability criterion for discrete-time systems, we draw the diagram of $\tilde{G}(e^{j\omega T})$ having the cyclic frequency ω as a parameter.

Theorem11.6.1

Assume that the transfer function $Z\{G(s)F(s)\}$ does not have any poles outside the unit circle. Then the closed-loop system is stable if the Nyquist diagram of $Z\{G(s)F(s)\}$ does not encircle the critical point $(-1,\quad j0)$ for $z=e^{jwT}$.

Clearly, the study of the stability of discrete-time closed-loop systems, as well as the design of discrete-time controllers, can be accomplished on the basis of the Nyquist diagrams, by extending the known techniques of the continuous-time systems as was done in the case of the Bode diagrams in the previous subsection.

11.6.3 Controller design using the root locus method

Let us consider the transfer function of the system shown in figure11.17. Recall that $G(s)=G_h(s)G_p(s)$. The closed-loop transfer function is

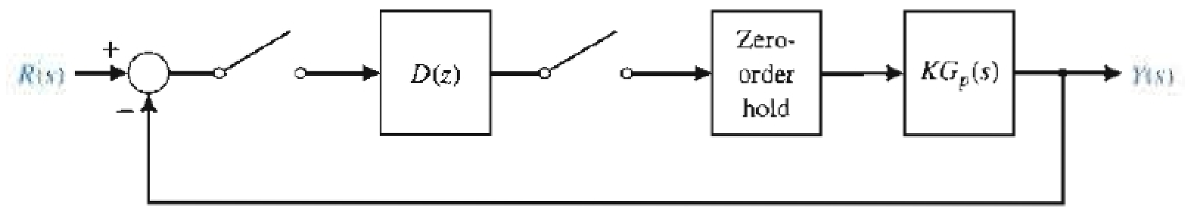


Figure11.17 Closed-loop control system with digital controller

$$\frac{Y(z)}{R(z)}=\frac{KG(z)D(z)}{1+KG(z)D(z)} \tag{11.152}$$

The digital controller design using root locus method have the following steps:

Step1. The root locus starts at the poles and progresses to the zeros in the z plane.

Step2. The root locus lies on a section of the real axis to the left of an odd number of poles and zeros.

Step3. The root locus is symmetrical with respect to the horizontal real axis.

Step4. The root locus may break away from the real axis and may reenter the real axis. The breakaway and entry points are determined from the equation with $z=\sigma$

$$K=-\frac{N(z)}{D(z)}=F(z) \tag{11.153a}$$

Then obtain the solution of $\dfrac{dF(\sigma)}{d\sigma}=0$.

Step5. Plot the locus of roots that satisfy

$$1+KG(z)D(z)=0$$

Or

$$|KG(z)D(z)|=1 \quad \text{and} \quad \angle G(z)D(z)=180^o \pm k360^o,\quad k=0,1,2\cdots \tag{11.153b}$$

The characteristic equation is

$$1+KG(z)D(z)=0 \tag{11.153c}$$

Equation(11.153c) is analogous to the characteristic equation for the s-plane analysis of $KG(s)$. Thus we can plot the root locus for the characteristic equation of the sampled system as K varies.

Example11.17 Root Locus of second-order system.

Consider the system shown in figure11.17 with $D(z)=1$ and $G_p(s)=1/s^2$. Then we can obtain

$$KG(z)=\frac{T^2}{2}\frac{K(z+1)}{(z-1)^2}$$

Let $T=\sqrt{2}$ and plot root locus. Then we have

$$KG(z)=\frac{K(z+1)}{(z-1)^2}$$

The pole and zeros are shown on the z-plane in figure11.18.

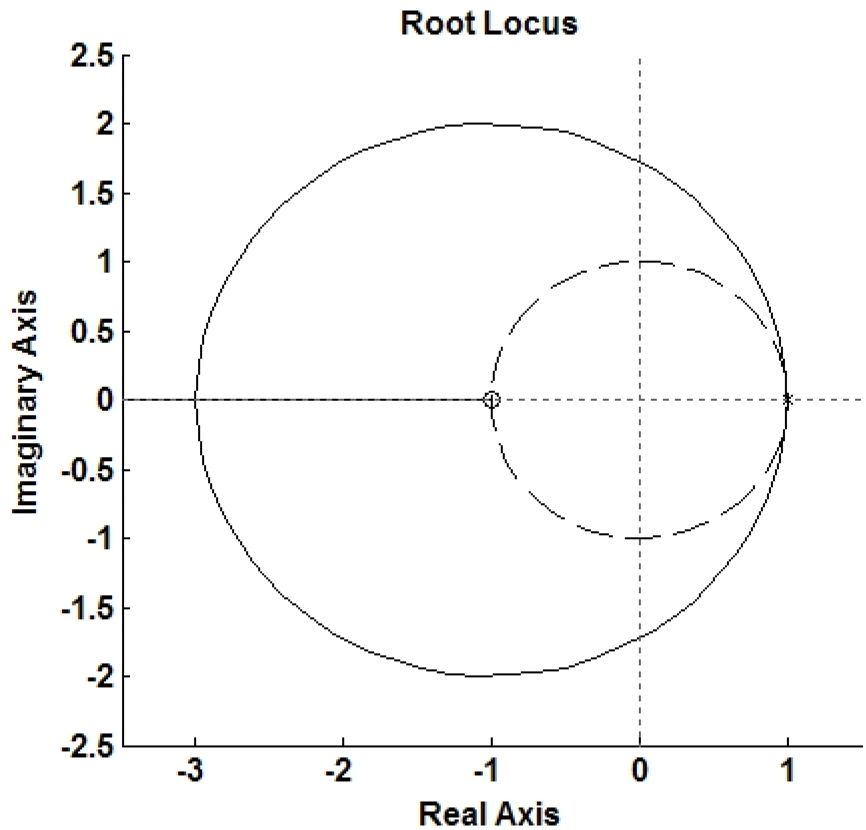


Figure11.18 Root locus of example11.17

The characteristic equation is

$$1+KG(z)=1+\frac{K(z+1)}{(z-1)^2}=0$$

Let $z=\sigma$, and solve for K to obtain the following expression

$$K = \frac{(\sigma - 1)^2}{\sigma + 1} = F(\sigma)$$

Order $\frac{dF(\sigma)}{d\sigma} = 0$, and we have

$$(\sigma - 1)(\sigma + 3) = 0$$

Solution is $\sigma_1 = 1$, $\sigma_2 = -3$

The locus leaves the two poles at $\sigma_1 = 1$ and reenter at $\sigma_1 = -3$, as shown in figure11.17. The unit circle is also described by broken line shown in figure11.17. The system always has two roots outside the unit circle and is always unstable for all $K > 0$.

We now turn to design a digital controller $D(z)$ to achieve a specified response utilizing a root locus method. Here, select a controller

$$D(z) = \frac{z - a}{z - b}$$

We then use $(z - a)$ to cancel one pole at $G(z)$ that lies on the positive real axis of the z-plane. Then we select $(z - b)$ so that the locus of the compensated system will give a set of complex roots at a desired point within the unit circle on the z plane.

For $D(z) = 1$, we know that the system in example11.17 is unstable. Here, we select

$$D(z) = \frac{z - a}{z - b}$$

Therefore, $KG(z)D(z) = \frac{K(z + 1)(z - a)}{(z - 1)^2 (z - b)}$

If select $a = 1$ and $b = 0.2$, we have

$$KG(z)D(z) = \frac{K(z + 1)}{(z - 1)(z - 0.2)}$$

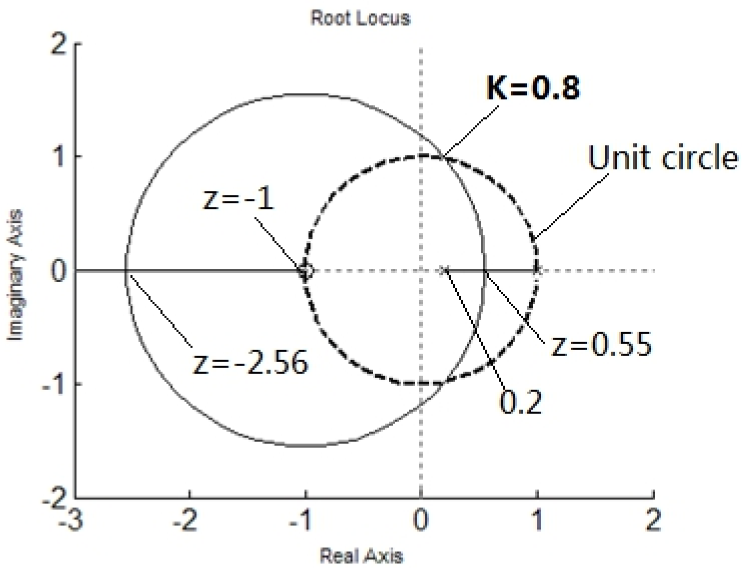


Figure11.19 Root locus for $a = 1$, $b = 0.2$

Using the equation for $F(\sigma)$, we obtain the entry point as $z = -2.56$, as shown in figure11.19. The root locus is on the unit circle at $K = 0.8$. Thus, the system is stable for $K < 0.8$.

If the system performance were inadequate, we may improve the root locus by select $a = 1$ and $b = -0.98$ so that

$$KG(z)D(z) = \frac{K(z+1)}{(z-1)(z+0.98)} \approx \frac{K}{z-1}$$

Then the root locus would lie on the real axis of the z plane. When $K = 1$, the root of the characteristic equation is at the origin, and $T(z) = 1/z = z^{-1}$. Then the response of the sampled system(at the sampling instants) is the input step delayed by one sampling period.

We can draw lines of constant ζ on the z plane. The mapping between the s plane and the z plane is obtained by the relation $z = e^{sT}$. The curve of constant ζ on the s plane are radial curves with

$$\frac{\sigma}{\omega} = -\tan\theta = -\tan\left(\sin^{-1}\zeta\right) = -\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Because $s = \sigma + j\omega$, we have

$$z = e^{\sigma T} e^{j\omega T} \quad z = e^{\sigma T} e^{j\omega T} = e^{\sigma T} \angle(\omega T)$$

Where

$$\sigma = -\frac{\zeta}{\sqrt{1-\zeta^2}}\omega$$

Commonly, damper coefficient ζ for may specifications is $1/\sqrt{2}$, then $\sigma = -\omega$.

11.7 Digital PID Controller

In discrete-time systems, the PID controller is widely used in practice. This section is devoted to the study of discrete-time PID controllers. We will first study separately the proportional(P), the integral(I) and the derivative(D) controller and, subsequently, the composite PID controller.

11.7.1 The proportional controller

For continuous-time systems, the proportional controller can be described by the following expression

$$u(t) = K_p e(t) \tag{11.154a}$$

Therefore, the transfer function of the proportional controller can be gained.

$$G_c(s) = K_p \tag{11.154b}$$

For discrete-time system, the proportional controller can be described by the relation

$$u(k) = K_p e(k) \tag{11.154c}$$

Hence,

$$G_c(z) = K_p \tag{11.154d}$$

11.7.2 The integration controller

For continuous-time systems, the integral controller can be described by the following integral equation.

$$u(t) = \frac{K_p}{T_i}\int_0^t e(t)dt \tag{11.155a}$$

Therefore, the transfer function of the integral controller is

$$G_c(s) = \frac{K_p}{T_i s} \tag{11.155b}$$

Here, the constant T_i is called the integration time constant or reset constant. In the case of discrete-time systems, the integral equation(11.154a) is approximated by the difference equation.

$$\frac{u(k) - u(k-1)}{T} = \frac{K_p}{T_i} e(k) \tag{11.155c}$$

Or,

$$u(k) = \frac{K_p T}{T_i} e(k) + u(k-1)$$

Hence, the z transfer function of the integral controller yields

$$G_c(z) = \frac{K_p T}{T_i\left(1 - z^{-1}\right)} = \frac{K_p T z}{T_i (z-1)} \tag{11.155d}$$

11.7.3 The derivative controller

For continuous-time systems, the derivative controller can be described by the following integral equation.

$$u(t) = K_p T_d \dot{e}(t) \tag{11.156a}$$

And therefore

$$G_c(s) = K_p T_d s \tag{11.156b}$$

Here, the constant T_d is called the derivative time constant or rate time constant. In the case of discrete-time systems, the derivative equation(11.156a) is approximated by the difference equation.

$$u(k) = K_p T_d \frac{e(k) - e(k-1)}{T} \tag{11.157a}$$

Here, T is sampling period.

Hence,

$$G_c(z) = \frac{K_p T_d \left(1 - z^{-1}\right)}{T} = \frac{K_p T_d (z-1)}{Tz} \tag{11.157b}$$

11.7.4 The PID controller

Combine above three kinds of controller, we can get the PID controller. For continuous-time systems, the PID controller can be described by the following integral equation.

$$u(t) = K_p \left[e(t) + \frac{1}{T_i} \int_{t_0}^{t} e(t) dt + T_d \frac{de(t)}{dt} \right] \tag{11.158a}$$

Therefore, the transfer function can be confirmed as

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) \tag{11.158b}$$

In the case of discrete-time systems, the PID equation(11.158a) is approximated by the following difference equation.

$$u(k) = K_p \left[e(k) + \frac{T}{T_i} \sum_{i=0}^{k-1} e(i) + \frac{T_d}{T} \left(e(k) - e(k-1) \right) \right] \tag{11.159a}$$

Hence,

$$G_c(k) = K_p \left[1 + \frac{Tz}{T_i (z-1)} + \frac{T_d (z-1)}{Tz} \right] \tag{11.159b}$$

Merge equation(11.159b) and we can get

$$G_c(z) = K \left[\frac{z^2 - az + b}{z(z-1)} \right] \tag{11.160}$$

Here,

$$K = K_p\left(\frac{TT_i + T_dT_i + T^2}{T_iT}\right), \qquad a = \frac{T_iT - T_dT_i}{T_iT + T_dT_i + T^2}, \qquad b = \frac{T_dT_i}{TT_i + T_dT_i + T^2}$$

11.8 Steady-state Error

11.8.1 Steady-state error

There also exists steady-state error in discrete-time system. In this section, we briefly cover this subject. Before discussing this theme, two theorems are given here.

Theorem11.8.1: Initial value theorem

The initial value of sampled function and its z transformation function have the following relation:

$$f(0) = \lim_{z\to\infty} F(z) \tag{11.161}$$

Demonstration

The z-transform of $f(kT)$ may be written as

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots$$

Taking the limits of both sides of the above equation as $z \to \infty$, we immediately achieve above relation(11.162).

Theorem11.8.2: Final value theorem

The final value of sampled function and its z transformation function have the following relation:

$$\lim_{k\to\infty} f(kT) = \lim_{z\to 1}\left(1 - z^{-1}\right)F(z) \tag{11.162}$$

Demonstration

Consider the z-transform of $f(kT+T) - f(kT)$.

$$Z[f(kT+T) - f(kT)] = \lim_{n\to\infty}\sum_{k=0}^{n}[f(kT+T) - f(kT)]z^{-k}$$

Then we obtain

$$zF(z) - zf(0) - F(z) = \lim_{n\to\infty}\sum_{k=0}^{n}[f(kT+T) - f(kT)]z^{-k}$$

Or,

$$\left(1 - z^{-1}\right)F(z) - f(0) = \lim_{n\to\infty}\sum_{k=0}^{n}[f(kT+T) - f(kT)]z^{-k-1}$$

Taking the limits of both sides of the above equation as $z \to \infty$, we immediately achieve

$$\lim_{z\to 1}\left(1-z^{-1}\right)F(z)-f(0)=\lim_{n\to\infty}\sum_{k=0}^{n}\left[f(kT+T)-f(kT)\right]z^{-k-1}$$
$$=\lim_{n\to\infty}\left\{\left[f(T)-f(0)\right]+\left[f(2T)-f(T)\right]+\cdots+\left[f(nT+T)-f(nT)\right]\right\}=-f(0)+f(\infty)$$

Hence,

$$f(+\infty)=\lim_{k\to\infty} f(kT)=\lim_{z\to 1}\left(1-z^{-1}\right)F(z)$$

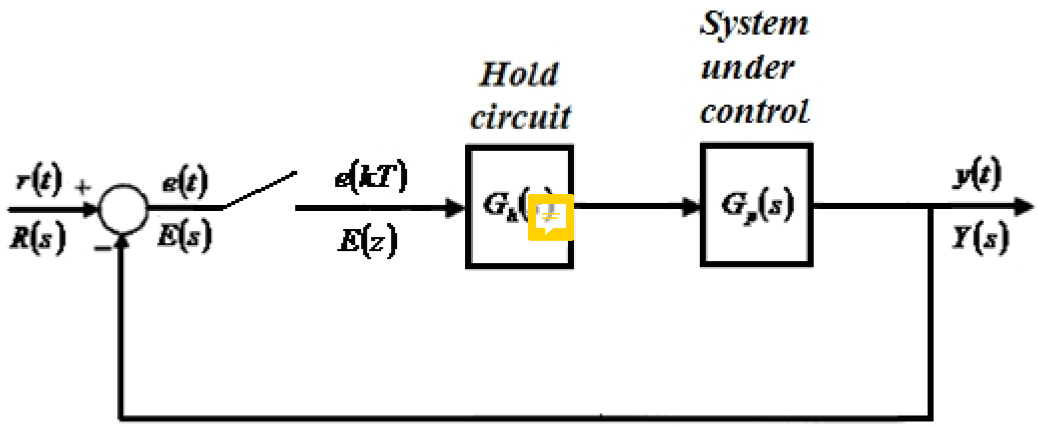


Figure11.20 Unit feedback closed-loop discrete-time system

Let's consider the unity feedback discrete-time closed-loop system shown in figure11.20. Assume that the system under control is stable, that's the final value of discrete-time system exists. Define

$$\hat{G}(z)=Z\left\{G_h(s)G_p(s)\right\}=Z\left\{\left(\frac{1-e^{-Ts}}{s}\right)G_p(s)\right\}=\left(1-z^{-1}\right)Z\left\{\frac{G_p(s)}{s}\right\}$$

Then, the closed-loop transfer function $H(z)$ will be

$$H(z)=\frac{\hat{G}(z)}{1+\hat{G}(z)} \tag{11.163}$$

The error $E(z)$ is given by

$$E(z)=R(z)-\hat{G}(z)E(z)$$

Hence

$$E(z)=\frac{R(z)}{1+\hat{G}(z)} \tag{11.164}$$

Applying final value theorem to steady state error, we can get that the steady-state error of error sampled function $e(kT)$, denoted as e_{ss}, is defined as

$$e_{ss}=\lim_{k\to\infty} e(kT)=\lim_{z\to 1}\left(1-z^{-1}\right)E(z) \tag{11.165}$$

If equation(11.164) is substituted into equation(11.165), then we have

$$e_{ss} = \lim_{z \to 1}\left(1 - z^{-1}\right)E(z) = \lim_{z \to 1}\left(1 - z^{-1}\right)\frac{R(z)}{1 + \hat{G}(z)} \tag{11.166}$$

11.8.2 Steady-state error for common functions

In this section, we will consider three particular input excitation: namely, the step function, the ramp function, and the acceleration function for they are commonly used in industry or scientific research.

1. Step function

In this case, $r(t) = 1$ or $r(kT) = 1$, from the definition of z transformation, there is

$$R(z) = Z\{r(kT)\} = Z\{1\} = \frac{1}{1 - z^{-1}}$$

Replacing above expression into equation(11.166) yields

$$e_{ss} = \lim_{z \to 1}\left\{\left(1 - z^{-1}\right)\left[\frac{1}{1 + \hat{G}(z)}\right]\left(\frac{1}{1 - z^{-1}}\right)\right\} = \lim_{z \to 1}\left[\frac{1}{1 + \hat{G}(z)}\right] = \frac{1}{1 + K_p}, \quad K_p = \lim_{z \to 1}\hat{G}(z) \tag{11.167}$$

Here, K_p is called the *position error constant*.

2.Ramp function(also called slope function)

In this case, $r(t) = t$ or $r(kT) = kT$, in the same way, we have

$$R(z) = Z[r(kT)] = Z\{kT\} = \frac{Tz^{-1}}{\left(1 - z^{-1}\right)^2}$$

Applying above equation into equation(11.166) yields

$$e_{ss} = \lim_{z \to 1}\left[\left(1 - z^{-1}\right)\frac{R(z)}{1 + \hat{G}(z)}\right] = \lim_{z \to 1}\left[\left(1 - z^{-1}\right)\frac{1}{1 + \hat{G}(z)} \times \frac{Tz^{-1}}{\left(1 - z^{-1}\right)^2}\right] = \lim_{z \to 1}\frac{T}{\left(1 - z^{-1}\right)\hat{G}(z)} = \frac{1}{K_v} \tag{11.168}$$

$$K_v = \lim_{z \to 1}\left[\frac{\left(1 - z^{-1}\right)\hat{G}(z)}{T}\right]$$

Here, K_v is called the *velocity error constant*.

3.Acceleration

In this case, $r(t) = \frac{t^2}{2}$ or $r(kT) = \frac{1}{2}(kT)^2$, applying z transformation yields

$$R(z) = Z\{r(kT)\} = Z\left\{\frac{1}{2}(kT)^2\right\} = \frac{T^2\left(1 + z^{-1}\right)z^{-1}}{2\left(1 - z^{-1}\right)^3}$$

Substituting the $R(z)$ in equation(11.166) yields

$$e_{ss} = \lim_{z\to 1}\left\{\left(1-z^{-1}\right)\left[\frac{1}{1+\hat{G}(z)}\right]\left[\frac{T^2\left(1+z^{-1}\right)z^{-1}}{2\left(1-z^{-1}\right)^3}\right]\right\} = \lim_{z\to 1}\left[\frac{T^2}{\left(1-z^{-1}\right)^2\hat{G}(z)}\right] = \frac{1}{K_a} \tag{11.168}$$

$$K_a = \lim_{z\to 1}\left[\frac{\left(1-z^{-1}\right)^2\hat{G}(z)}{T^2}\right]$$

Here, K_a is called *acceleration error constant.*

When the open-loop transfer function $\hat{G}(s)$ of a discrete-time system has the following form

$$\hat{G}(z) = \frac{1}{(z-1)^j}\left[\frac{a(z)}{b(z)}\right] \tag{11.169}$$

such discrete-time system is called a type j system, where $a(z)$ and $b(z)$ are z polynomials which do not involve the term $(z-1)$.

11.9 Design of Mechanical Worktable Motion Control

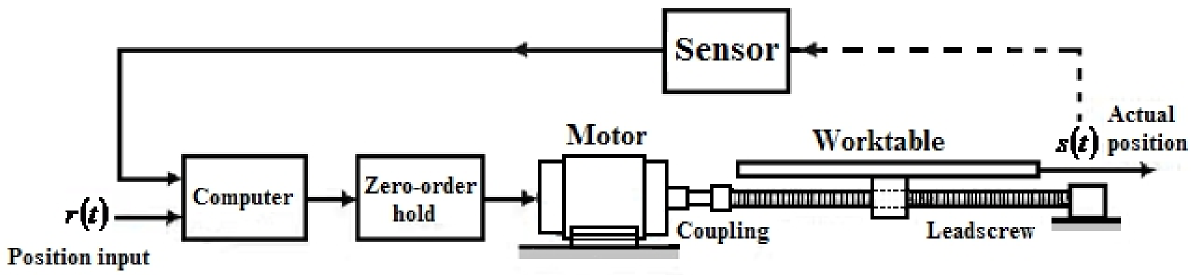


Figure11.21 Mechanical worktable motion control system

Worktable motion position control system has an very important role in modern manufacturing system. System controls the worktable position at a certain location. Here, we assume that the worktable is activated in each axis by a motor and lead screw, as shown in figure11.21. For figure11.21, we only consider the x-axis and check the motion control for a position feedback system, which block diagram is shown in figure11.22. The design goal is to obtain a fast response with a rapid rise time and settling time to a step common while not exceeding an overshoot of 5%.

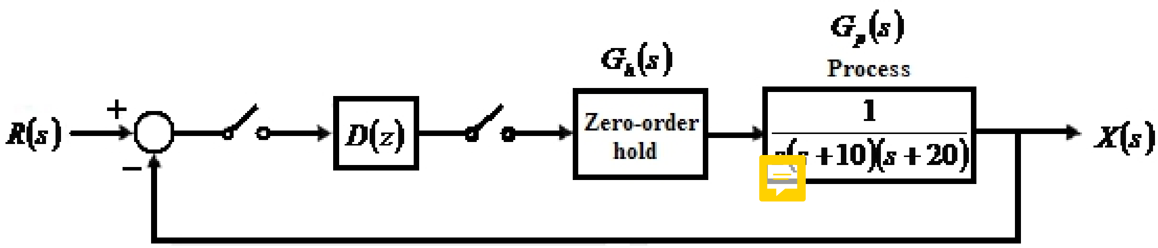


Figure11.22 Block diagram of Mechanical worktable motion control system

To configure the system, we choose a power amplifier and motor so that the system is described by the block diagram in figure11.23. Obtaining the transfer function of the motor and power amplifier, we have

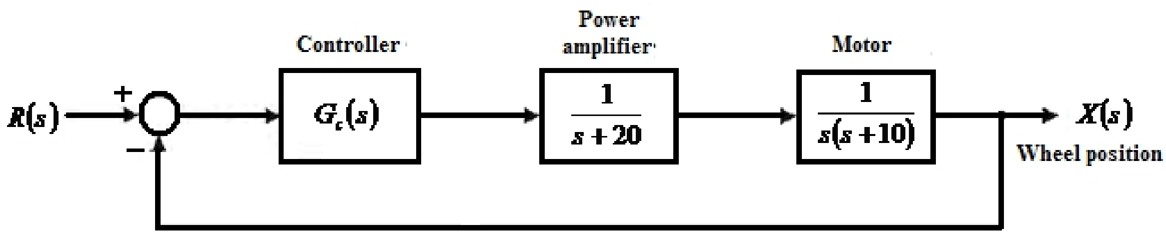
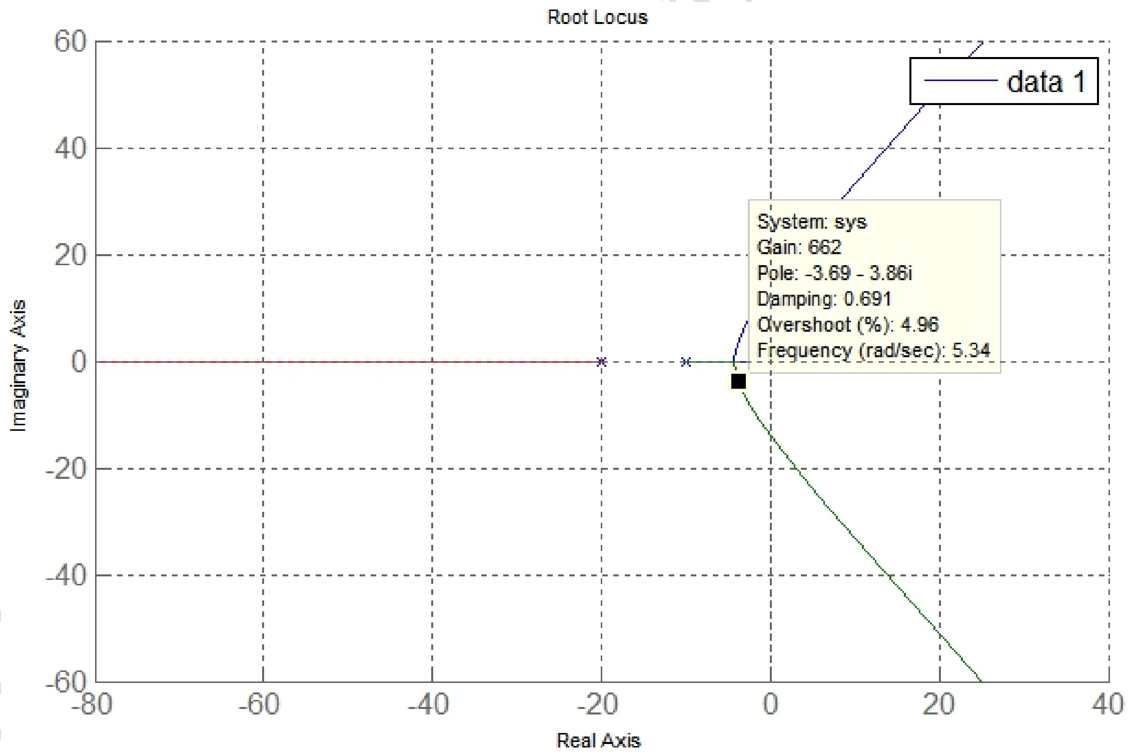


Figure11.23 Block diagram model of wheel control in worktable

$$G_p(s)=\frac{1}{s(s+10)(s+20)}$$

In mechanical worktable control system, we obtain $D(s)$ from $G_c(s)$. In the following we confirm the z transform of given function $G_p(s)$ using the root locus method. Firstly, we select the controller as a simple gain in order to determine the response that can be achieved without a compensator. Plotting the root locus shown in figure11.24, we find that in the vicinity of $K=662$, the dominant complex roots have a damping ratio of 0.691, and we expect a 5% overshoot.



Figurer11.24 Bode graph of $G_p(s)$

From the overshoot5%, we can calculate the damping ratio $\zeta=0.6901$.

Then we introduce the digital controller

$$G_c(s)=K\frac{s+a}{s+b}$$

We will select the zero at $s=-11$ so that the complex roots near the origin dominate.

Utilizing the lead-lag compensator in classical method, we find that we require the pole at $s=-62$. Evaluating the gain at the roots, we find that $K=8000$. Now select sampling period $T=0.01\sec$, we have

$$G_c = 8000\frac{s+11}{s+62}$$

Then

$$D(z) = C\frac{z+A}{z+B}$$

Where

$$A=-e^{-11T}=-0.8958 \quad \text{and} \quad B=-e^{-62T}=-0.5379$$

Now we can get

$$C\left(\frac{1+A}{1+B}\right)=C\left(\frac{1-0.8958}{1-0.5379}\right)=8000\left(\frac{0+11}{0+62}\right), \text{namely } C=6293$$

Hence, $D(z)=6293\left(\dfrac{z-0.8958}{z-0.5379}\right)$.

The performance comparison between proportional controller and $D(z)$ controller is listed out in table11.6.

Table11.6 Performance comparison between (1) controller and (2) controller

Compensator $G_c(s)$	K	Overshoot(%)	Settling time(sec)	Rising time(Sec)
(1) K	700	5	1.12	0.4
(2) $K(s+11)/(s+62)$	8000	5	0.6	0.25

Obviously, the performance of $D(z)$ controller is better than that of proportional controller K. Using this $D(z)$, we can get a desired response very similar to that obtained from the continuous system model.

Chapter summary

Digital control systems are described and analyzed basically, especially for linear time-invariant discrete-time system and sampled-data systems. Furthermore, the stability and its application examples of digital control system are explained. Furthermore, the controllability and observability and optimal control are respectively narrated and explained for digital control system. Besides these, how to design discrete-time controller is provided for readers, and digital PID controller are also narrated and deduced. Finally, steady-state error of modern digital control system are explained and exampled with respect to the concept in the classical control. In this chapter, readers need to know the common block diagram, stability, controllability and observability and optimal control on modern control system; especially, steady-state error needs to be understood and grasped.

Related Readings

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[10]Konstantinos G. Papadopoulos, Nikolaos D. Tselepis, Nikolaos I. Margaris. *"Type-III closed loop control systems-Digital PID controller design"*, Journal of Process Control, Volume 23, Issue 10, November 2013, Pages 1401-1414.

Review Questions

11.1. Please talk about the concept of digital control and your comprehension.

11.2. Which methods can be utilized to judge the stability of digital control system?

11.3. Please talk about the controllability difference between linear continuous-time system and linear digital control system.

11.4. Please give out the formula of judging the controllability of digital control system.

11.5. What is the observability criterion or formula of a digital control system?

11.6. How is linear quadratic regulator expressed in optimal control?

11.7. Please talk about digital PID controller and the implementation algorithm you know.

11.8. What is steady-state error? Please give out some examples you know to demonstrate the role of steady-state error.

11.9. Please talk about the expression and implementation of the steady-state quadratic regulator.

Problems

Problem11.1. A discrete-time system is described by the following state-space equations

$$x(k+1)=Ax(k)+Bu(k),\quad y(k)=Cx(k)+Du(k)$$

Where

$$A=\begin{bmatrix}0 & 1\\ -3 & -2\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix},\quad C=\begin{bmatrix}1\\ 1\end{bmatrix},\quad D=0$$

Please find the transition matrix, the transfer function, and the response of the system when the input is the unit step sequence.

Problem 11.2. Please find the following transfer function of the systems described by the difference equations.

(1) $y(k+2)+2y(k+1)+4y(k)=u(k+1)+2u(k)$;

(2) $y(k+2)-2y(k)=u(k)$.

Problem11.3.Consider the following system,

$$\begin{aligned}&x(k+1)=Ax(k)+Bu(k),\quad x(0)=[0\quad 0]^T\\ &y(k)=Cx(k)\end{aligned}$$

where

$$A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},\quad B=\begin{bmatrix}1\\ 0\end{bmatrix},\quad C=[1,\quad 0]$$

Please find a control sequence, if it exists, that can drive the system to desired final system state $\xi=[2\quad 2.3]^T$.

Problem11.4. Check the stability of the system described by

$$x_1(k+1)=x_1(k)+x_2(k),\quad x_2(k+1)=2x_1(k)+x_2(k)+u(k),\quad y(k)=x_1(k)$$

If the system is unstable, use the output feedback law $u(k)=-gy(k)$ to stabilize it.

Determine the range of values of a suitable parameter g.

Problem11.5. A discrete-time system can be described by equations