Controllability and Observability for Linear System

Controllability and observability are very important two conceptions of control system in modern control theory; hence they will be narrated and demonstrated in this chapter. This chapter will discuss the controllability for linear system and its judgment criterion, observability for linear system and its judgment criterion. And then structural decomposition and the minimal realization of transfer function matrix will be also discussed.

Objectives

By the end of this chapter, you should be able to:

- Know the basic conceptions of controllability and observability.
- Judge the controllability and observability of linear time-invariant system.
- Decompose the structure of linear system.
- Make the controllability and observability of state space expression standardize.
- Comprehend the minimal realization of transfer function matrix.

5.1 Introduction of Controllability and Observability

The basic conceptions of controllability and observability are put forward firstly by Kalman. In multi-variable optimal control system, these two notions are very important. According to these two conception, we can find the whole condition where optimal control could be existent. In control engineering, people usually pay attention to two problems: 1) after a input signal is input into a given system, can system state be converted from original state to predicted state in a finite time?2) by observing system output for a certain time, is original state judged? Such is the problem of system controllability and observability. If a control system is not controllable, optimal control will not be realized successfully. If a control system is not observable, state observer will not be designed and assigned.

In this section, we will firstly introduce linear correlation and linear independence, and then give out the definition and condition of system controllability and observability.

5.1.1 Linear correlation and linear independence

If n numbers constitute a sequencing array, such array is called n-dimension vector.

n-dimension vector can be written into row vector, also into column vector, namely

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{5.1}$$

Or, written into the following form:

$$x = [x_1, \quad x_2, \quad \cdots, \quad x_n]^T \tag{5.2}$$

Vector $x_1, \quad x_2, \quad \cdots, \quad x_n$ is that the following equation is found

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0 \tag{5.3}$$

If all coefficients in equation(5.3) are zero, vector $x_1, \quad x_2, \quad \cdots, \quad x_n$ is linear correlation. Otherwise, vector $x_1, \quad x_2, \quad \cdots, \quad x_n$ is linear independence.

For example, the following vector

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These three vectors are linear independence for all coefficients in equation(5.3) are zero. If a vector x_i can be expressed by the linear combination of other vectors $x_1, \quad x_2, \quad \cdots, \quad x_n$ in terms of the form in equation(5.3), namely,

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j x_j \tag{5.4}$$

Then vectors $x_1, \quad x_2, \quad \cdots, \quad x_n$ are called linear correlation. For the following vector,

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Above three vectors are linear correlation for $x_3 = x_1 + x_2$.

As we have known, modern control system is found on the basis of state space. State equation is used to depict the relationship between input signal $u(t)$ and state $x(t)$; the output equation is employed to describe the output change that caused by system state. Controllability and observability could sufficiently demonstrate the control ability of input signal $u(t)$ to system state $x(t)$ and the exhibition ability of system output $y(t)$ to system state $x(t)$. However classical control theory is only used to discuss the control ability of input signal to output signal. Controllability and observability are very important in modern control theory. Hence, it’s necessary for us to discuss the controllability and observability of system.

5.1.2 Definition of controllability

Controllability is used to judge the control ability of input signal $u(t)$ to system state $x(t)$, which is not related to system output $y(t)$. Hence, we only study the controllability of input signal to system state.

1) The definition of controllability for linear continuous time-invariant system

Assume that linear continuous time-invariant system is as follows

$$\dot{x} = Ax + Bu \tag{5.5}$$

For any system state $x(t_0)$ and another system state $x(t_1)$, if there is a finite time $(t_0, \ t_1)$ and a sectional continuous input signal $u(t)$, they could successfully make system state $x(t_0)$ be transformed into another system state $x(t_1)$ in a finite time $(t_0, \ t_1)$. Such system state is controllable, otherwise it is not controllable. If all system states are controllable, such system state is completely controllable, which is briefly called that such system is controllable.

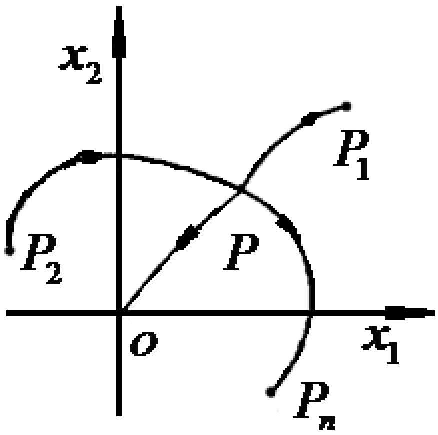

Figure5.1 Phase plane

Above definition can be demonstrated in a phase plane shown in figure5.1. If point P in phase plane can be converted into any a pointed state $P_1, \ P_2, \ \cdots, \ P_n$ under the input signal, point P in phase plane is a controllable state. If controllable state could be filled in the whole phase plane, that’s to say, for any original state a corresponding input $u(t)$ could be found in a finite time, original system state could be successfully transformed into any pointed point in phase plane. Such system is called state complete controllability.

Obviously, the controllability of some a state in system is very different from state complete controllability.

The following points need to be demonstrated.

- In linear time-constant system, at original time $t_0 = 0$, original state is $x(0)$; any final state $x(t_f)$ can be pointed to be zero state, namely,

$$x(t_f) = 0$$

- $x(0)$ can be also pointed to be zero state, and $x(t_f)$ is any final state. That's to say, if there is an unconfined control input signal $u(t)$ in a finite time $(t_0,\ \ t_1)$, it could make system state $x(t)$ be converted into any state $x(t_f)$ at the beginning of zero state. Such case is called state reach-ability. For linear time-constant system, controllability and reach-ability is interchanged.
- When we discuss controllability of system, control role in theory is unconfined, ~~which value is not unique.~~

2) The definition of controllability for linear continuous time-variant system

Assume that linear continuous time-variant system is as follows:

$$\dot{x} = A(t)x + B(t)u \tag{5.6}$$

The definition of controllability for linear continuous time-variant system is the same as that for linear continuous time-invariant system; but matrices $A(t)$ and $B(t)$ are time-variant matrices, not constant coefficient matrices.The transform of system state $x(t)$ is related to the original time t_0 selected. Hence, the emphasis in linear time-variant system is that system is controllable at $t = t_0$.

3) The definition of controllability for linear discrete time-invariant system

Linear discrete time-variant system is as follows:

$$x(k+1) = Ax(k) + Bu(k) \tag{5.7}$$

Function $u(k)$ is input scalar function, which is constant in $(k,\ \ k+1)$. Its controllability is defined as follows:

In finite time $t \in [0,\ \ nT]$, if there is a sequence $u(0),\ \ \cdots,\ \ u(n-1)$, this sequence could make system with original state $x(0)$ be successfully converted into any final state $x(n)$. It's thoughtful that system state is completely controllable.

Now let's consider the specific parallel form shown in Figure5.2(a). To control the pole position of closed-loop system, we are saying implicitly that the control signal, u, can control the behavior of each state variable in x. If any one of the state variables can not be controlled by the control u, then we can not place the poles of the system where we desire. For example, in Figure5.2(b), if x_1, were not controlled by the control signal and if x_1 also exhibited an unstable response due to a nonzero initial condition, there would be no way to effect a state-feedback design to stabilize x_1; x_1 would perform in its own way regardless of the control signal, u. Thus in some systems, a state-feedback design is not possible.

Hence, we can know the following conclusions from above examples:

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise, the system is uncontrollable.

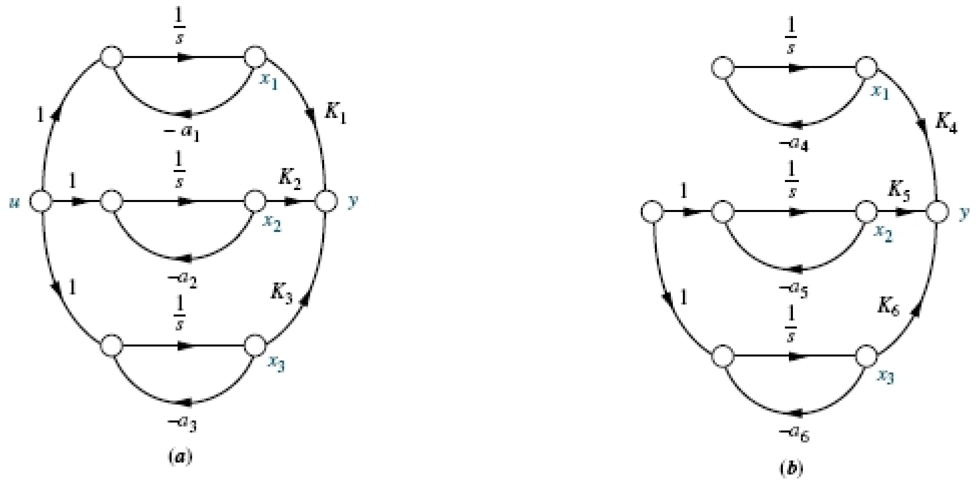

Figure5.2 Comparison between a controllable and an uncontrollable systems

5.1.2 Definition of observability

Feedback control form are usually adopted by plenty of control systems.In modern control theory, feedback information is composed of state variables; but not all state variables are observable in physics. Hence people started to study whether all state variable information could be acquired by detecting system output. If every state variable could affect system output, system state would be observable. If some state in all states does not affect system output, such system will be not observable. Of course, the information of some state is not acquired from given system output.

System observability demonstrates the ability of that system output reflects system state, which is not related to input signal. That's to say, observability is a property of the coupling between system state and system output. Thus observability involves the matrices A and C.Hence, when system observability is analyzed, homogeneous state equation and output equation are only considered, namely,

$$\dot{x} = Ax, \quad x(t_0) = x_0 \tag{5.8a}$$

$$y = Cx \tag{5.8b}$$

A linear system is said to be observable at t_0 if $x(t_0)$ can be determined from the output function $y_{[t_0,\ t]}$ (or output sequence) for $t_0 \le t$, where t is some finite time belonging to real. If this is true for all t_0 and $x(t_0)$, the system is said to be completely observable.

Clearly the observability of a system will be a major requirement in filtering and state estimation or reconstruction problems. In many feedback control problems, the controller must use output variables y rather than the state vector x in forming the feedback signals. If the system is observable, then system output y contains sufficient information about the internal states so that most of the power of state feedback can still be realized. A more complicated controller is needed to achieve these results. This is discussed fully in Chapter11.

The following points need to be demonstrated in terms of above observability definition.

- What observability expresses is the ability of that system output $y(t)$ reflects system state $x(t)$ .Since the system output $y(t)$ caused by input signal $u(t)$ could be calculated, we can order $u(t)=0$ and only consider homogeneous state equation and output equation in equation(5.8).
- On considering system output equation, we can find that solving state $x(t)$ is very simple if dimension of system output $y(t)$ is equal to that of system state $x(t)$, $m=n$ and further if matrix C is non-singular matrix. Namely, system state could be expressed into the following form:

$$x(t)=C^{-1}y(t) \tag{5.9}$$

- So long as the initial state is ensured, instant state of system could be gained in terms of given input signal $u(t)$ and state equation.

$$x(t)=\phi(t-t_0)x(t_0)+\int_{t_0}^{t}\phi(t-\tau)Bu(\tau)d\tau \tag{5.10}$$

5.2 Controllability of Linear System

Now we continue to explore the controllability from another viewpoint: that of the state equation itself. When the system matrix is diagonal for a linear system, as it is for the parallel form, it is apparent whether or not the system is controllable.For example, the state equation in Figure5.2(a) is the following:

$$\dot{x}=\begin{bmatrix}-a_1 & 0 & 0\\ 0 & -a_2 & 0\\ 0 & 0 & -a_3\end{bmatrix}x+\begin{bmatrix}1\\1\\1\end{bmatrix}u \tag{5.11}$$

Or,

$$\dot{x}_1=-a_1x_1+u \tag{5.12a}$$

$$\dot{x}_2=-a_2x_2+u \tag{5.12b}$$

$$\dot{x}_3 = -a_3 x_3 + u \quad (5.12c)$$

Since each of equations(5.12) is independent from the rest, the control signal u affects each of the state variables. This is controllability from another perspective. Now let us look at the state equations for the system of Figure5.2(b):

$$\dot{x} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \quad (5.13)$$

Or,

$$\dot{x}_1 = -a_4 x_1 \quad (5.14a)$$

$$\dot{x}_2 = -a_5 x_2 + u \quad (5.14b)$$

$$\dot{x}_3 = -a_6 x_3 + u \quad (5.14c)$$

From the state equation in (5.13) or (5.14), we see that the state variable x_1 is not controlled by the control signal u. Thus, the system is said to be uncontrollable.

In summary, a system with distinct eigenvalues and a diagonal system matrix is controllable if the input coupling matrix B (system input matrix) does not have any rows that are zero.

For linear system, let's explore the judgment criterion of system controllability matrix.

5.2.1 The first criterion of system controllability

Test for the controllability that we have so far explored can not be used for representations of the system other than the diagonal or parallel form with distinct eigenvalues. The problem of visualizing the controllability gets more complicated if the system has multiple poles, even though it is represented in parallel form.Furthermore, one can not determine controllability by inspection for systems that are represented in parallel form. In other forms, existence of paths from the input to the state variables is not a criterion for controllability since the equations are not decoupled.

In order to determine the system controllability, or alternatively, to design for a feedback for a plant under any representation or choice of state variables, a matrix is derived that must have a particular property if all state variables are controllable by the plant input, u . We now state the requirement for controllability, including form, property, and name of the matrix.

The first criterion of system controllability (Contraollability judgment criterion): An nth-order plant whose state equation is the following is completely controllable

$$\dot{x} = Ax + Bu \quad (5.15)$$

if the rank of matrix S_c is n,

$$rankS_c = rank\begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix} = n \tag{5.16}$$

where matrix S_c is called the controllability matrix. Here, number n is system dimension. Equation(5.16) is the first criterion of system controllability.

Example5.1 Please judge the state controllability of the following system.

$$\dot{x} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -2 & 0 \\ 1 & 3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Solution: System matrix A and input matrix B are as follows

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -2 & 0 \\ 1 & 3 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 2 & -8 \\ 0 & 0 & 0 \\ 1 & -3 & 11 \end{bmatrix}$$

$$rankS_c = 2 < n = 3$$

Hence, system is not completely controllable.

Example5.2 Please judge the state controllability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Solution: System matrix A and input matrix B are as follows

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 11 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$rankS_c = 3 = n$$

Hence, system is completely controllable.

Example5.3 Please judge the state controllability of the following system.

$$\dot{x} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} u$$

Solution: System matrix A and input matrix B are as follows

$$A=\begin{bmatrix}1&3&2\\0&2&0\\0&1&3\end{bmatrix}\qquad B=\begin{bmatrix}2&1\\1&1\\-1&-1\end{bmatrix}$$

$$S_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}2&1&3&2&5&4\\1&1&2&2&4&4\\-1&-1&-2&-2&-4&-4\end{bmatrix}$$

Number in the second row is proportional to that in the second row of judgment matrix S_c, $rankS_c=2<n=3$; hence system is not completely controllable.

In Multi-Input Multi-Output system, judgment matrix S_c becomes more complicate; and ensuring the rank of judgment matrix will become very difficult. The multiplication $S_c\bullet S_c^T$ of judgment matrix S_c and its transposition matrix S_c^T is a square matrix, and furthermore the non-singularity of matrix $S_c\bullet S_c^T$ is equal to that of matrix S_c; hence when calculating the matrix which rows are less than its columns, people usually make use of the relationship $rankS_c=rank\left(S_c\bullet S_c^T\right)$ to confirm the rank of judgment matrix S_c.

Example5.4 Please judge the state controllability of the following system.

$$\dot{x}=\begin{bmatrix}1&2&1\\0&1&0\\1&0&3\end{bmatrix}x+\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}u$$

Solution: System matrix A and input matrix B are as follows

$$A=\begin{bmatrix}1&2&1\\0&1&0\\1&0&3\end{bmatrix}\qquad B=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$AB=\begin{bmatrix}1&2&1\\0&1&0\\1&0&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&2\\0&1\\1&0\end{bmatrix},\quad A^2B=\begin{bmatrix}1&2&1\\0&1&0\\1&0&3\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&0\\1&0&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}=\begin{bmatrix}2&4\\0&1\\4&2\end{bmatrix}$$

$$S_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}1&0&1&2&2&4\\0&1&0&1&0&1\\0&0&1&0&4&2\end{bmatrix}\qquad S_c\bullet S_c^T=\begin{bmatrix}26&6&17\\6&3&2\\17&2&21\end{bmatrix}$$

Obviously, the rank of matrix $S_c\bullet S_c^T$ is non-singular; hence judgment matrix S_c is full rank, and system is controllable.

Example5.5 Please judge the state controllability of the following system.

$$x(k+1)=\begin{bmatrix}1&0&0\\0&2&-2\\-1&1&0\end{bmatrix}x(k)+\begin{bmatrix}1\\0\\1\end{bmatrix}u(k)$$

Solution: System matrix A and input matrix B are as follows

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -2\\ -1 & 1 & 0\end{bmatrix} \qquad B=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$S_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 1 & -1 & -3\end{bmatrix}=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & -2 & -4\end{bmatrix}$$

$$=rank\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -2\\ 0 & 0 & -2\end{bmatrix}=3=n$$

Hence, system is completely controllable.

5.2.2 The second criterion of system controllability

If system matrix A is diagonal matrix, we will have simpler controllability judgment criterion. Firstly, we should discuss whether the controllability of a system is kept after it is transformed into another form by linear transform. This answer is yes. Linear transform only changes state space of system, but the nature characteristic of system is not changed. Controllability is one of system nature characteristics.

Theorem The controllability of linear time-constant(continuous or discrete) system always keeps unchanged by any non-singular transform.

Demonstration: We assume that the state equation of linear time-constant continuous system is as follows

$$S: \quad \dot{x}=Ax+Bu \tag{5.17}$$

Equation(5.17) after a non-singular transformation $x=P\bar{x}$ is the following:

$$\bar{S}: \quad \dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u \tag{5.18}$$

In terms of controllability judgment criterion, the controllability matrix system S could be expressed as follows:

$$rankS_c=rank\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix} \tag{5.19}$$

According to the relationship between system S and system $\bar{S}$: $A=P\bar{A}P^{-1}$, $B=P\bar{B}$, we get

$$
\begin{aligned}
rankS_c &= rank\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \\
&= rank\begin{bmatrix} P\bar{B} & P\bar{A}P^{-1}P\bar{B} & P\bar{A}^2P^{-1}P\bar{B} & \cdots & P\bar{A}^{n-1}P^{-1}P\bar{B} \end{bmatrix} \\
&= rank\begin{bmatrix} P\bar{B} & P\bar{A}\bar{B} & P\bar{A}^2\bar{B} & \cdots & P\bar{A}^{n-1}\bar{B} \end{bmatrix} \\
&= rank\left\{ P\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix} \right\}
\end{aligned} \tag{5.20}
$$

For transform matrix P is an reversible matrix, that's to say, the rank of matrix P is full rank, equation(5.20) yields

$$
rank\,\bar{S}_c = rank\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix} = rankS_c \tag{5.21}
$$

Obviously, controllability judgment matrices of system S and system $\bar{S}$ have the same rank. Namely, if original system is controllable, new system by linear transform is also controllable. Hence, Through any non-singular linear transform, controllability of system always keeps unchanged.

The second criterion of system controllability for different eigenvalues: Assuming the state equation of a linear time-constant system is the following:

$$
\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \bar{x} + \bar{B}u \tag{5.22}
$$

The sufficient and necessary condition of complete controllability of system state is that input matrix $\bar{B}$ does not include the row which all elements are zero in diagonal canonical form gained after non-singular transform.

When we make use of this criterion to judge the diagonal standard of system, we should note the condition that there are different eigenvalues in system matrix. For example, please judge the following controllability of four given systems.

(1) $\dot{x} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} u$, this system is completely controllable.

(2) $\dot{x} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} u$, this system is not completely controllable.

(3) $\dot{x} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 7 & 4 \end{bmatrix} u$, this system is completely controllable.

(4) $\dot{x} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 7 & 4 \end{bmatrix} u$, this system is not completely controllable.

Some matrix with multiple eigenvalues can be also transformed into diagonal canonical form. For such system, above judgment criterion can not be utilized here. Then we have to use the first judgment criterion of system state controllability. For example, there is the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

System matrix A has multiple eigenvalues; hence input matrix B has not the rows which elements are zero, but it's easy to judge that this system is not completely controllable by the first method of controllability judgment criterion.

For the case that system matrix A has multiple eigenvalues, there is the following theorem.

The second method of controllability judgment criterion for multiple eigenvalues

If a linear time-constant system $\dot{x} = Ax + Bu$ has multiple eigenvalues each of which has its corresponding characteristic vector, the sufficient and necessary condition of complete controllability of system state is that all elements of the row in input matrix B corresponding to the last row of each Jordan small block J_i in system matrix A are not all zero in the following Jordan canonical form by the non-singular transform.

$$\dot{\bar{x}} = \begin{bmatrix} J_1 & \cdots & \cdots & 0 \\ \vdots & J_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & J_k \end{bmatrix} \bar{x} + \bar{B} u \tag{5.23}$$

The following systems are exampled to illustrate above theorem.

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u$, this system is completely controllable.

After observing this state equation, we can know that there is only a Jordan block. The element of the last row in input matrix $\bar{B}$ is 3 which is not zero; hence this system is completely controllable.

(2) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u$, this system is not completely controllable.

Here is only a Jordan block in system matrix $\bar{A}$,too. The element of the last row in input matrix $\bar{B}$ is 0; hence this system is not completely controllable.

(3) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} u$, this system is completely controllable.

Here is two Jordan blocks in system matrix $\bar{A}$. All elements of the last row in input matrix $\bar{B}$ corresponding to the Jordan block in system matrix $\bar{A}$ are not all zeros which are respectively $[0 \quad 2]$ and $[3 \quad 0]$; hence this system is completely controllable.

(4) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 3 & 2 \end{bmatrix} u$, this system is not completely controllable.

Obviously, there are two Jordan blocks in system matrix $\bar{A}$, too. The last row of the first Jordan block in input matrix $\bar{B}$ is $[0 \quad 0]$ where all elements are zero. Hence, this system is not completely controllable.

5.2.3 Output controllability and its judgment criterion

System controllability has been narrated in section5.22, then readers possibly ask whether system output can be controlled in state space. The correct answer is that system output could be controlled. We assume the generally linear time-invariant system is as follows:

$$\dot{x} = Ax + Bu \tag{5.24a}$$

$$y = Cx + Du \tag{5.24b}$$

Here, $x \in R^n$; $y \in R^l$; $u \in R^m$.

Definition of output controllablity For any an original output $y(t_0)$ and another output $y(t_1)$, if there exist a finite time $[t_0, \quad t_1]$ and a segment input function $u(t)$, they could make original output $y(t_0)$ be successfully transformed into predicted output $y(t_1)$; we think that such system is output controllable; otherwise such system output is not controllable.

Judgment criterion of output controllability The sufficient and necessary condition of output controllability is the following:

$$rankS_{ou} = rank[CB \quad CAB \quad CA^2B \quad \cdots \quad CA^nB \quad D] = l \tag{5.25}$$

Example5.6 Please judge the output controllability of the following system.

$$\dot{x}=\begin{bmatrix}-4 & 1\\ 2 & -3\end{bmatrix}x+\begin{bmatrix}1\\ 2\end{bmatrix}u\,,\quad y=\begin{bmatrix}1 & 0\end{bmatrix}x$$

Solution: Relative matrix of this system are as follows

$$A=\begin{bmatrix}-4 & 1\\ 2 & -3\end{bmatrix},\qquad B=\begin{bmatrix}1\\ 2\end{bmatrix},\qquad C=\begin{bmatrix}1 & 0\end{bmatrix},\qquad D=0$$

The output judgment matrix S_{ou} is the following:

$$rankS_{ou}=rank\begin{bmatrix}CB & CAB\end{bmatrix}=rank\begin{bmatrix}1 & 2\end{bmatrix}=1=l$$

Hence, this system output is controllable. However, we easily know that system state is not controllable.

Note that there is no common relationship between state controllability and output controllability.

5.2.4 Controllability of linear time-variant system

A continuous-time system with the representations $\{A(t),\ B(t),\ C(t),\ D(t)\}$ is considered in this section. The state equation of time-variant system is the following:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t) \tag{5.26}$$

For a given input function $u(t)$, the solution for the state at a fixed time t_1 is

$$x(t)=\phi(t_0,\quad t)x(t_0)+\int_{t_0}^{t}\phi(t,\quad \tau)B(\tau)u(\tau)d\tau \tag{5.27}$$

Here, $\phi(t_1,\quad t_0)$ is the state transition matrix.

Controllability judgment criterion

The sufficient and necessary condition of state complete controllability for linear time-variant system on the time interval $\lfloor t_0,\quad t_f \rfloor$ is that Gram matrix is non-singular which expression is the following.

$$G(t_0,\quad t_f)=\int_{t_0}^{t_f}\phi(t_0,\quad \tau)B(\tau)B^T(\tau)\phi^T(t_0,\quad \tau)d\tau \tag{5.28}$$

Demonstration: Sufficiency of equation(5.28)

If Gram matrix $G(t_0,\quad t_f)$ were to be non-singular, reversible matrix $G^{-1}(t_0,\quad t_f)$ would be existent. Select the following input function $u(t)$.

$$u(t)=-B^T(t)\phi^T(t_0,\quad t)G^{-1}(t_0,\quad t)x(t_0) \tag{5.29}$$

Now we need to analyze whether input function $u(t)$ makes original state $x(t_0)$ of system be converted into original point on the time interval $\lfloor t_0,\quad t_f \rfloor$. If the key is yes, it demonstrates that there exist an input function $u(t)$ expressed in equation(5.29) and that system is completely controllable.

As we know, the solution of equation(5.26) is equation(5.27), namely:

$$x(t)=\phi(t_0,\quad t)x(t_0)+\int_{t_0}^{t}\phi(t,\quad \tau)B(\tau)u(\tau)d\tau$$

Order $t=t_f$ and substitute equation(5.29) into above solution, we can get

$$\begin{aligned}
x(t_f)&=\phi(t_0,\quad t_f)x(t_0)+\int_{t_0}^{t_f}\phi(\tau,\quad t_f)B(\tau)u(\tau)d\tau\\
&=\phi(t_0,\quad t_f)x(t_0)-\int_{t_0}^{t_f}\phi(\tau,\quad t_f)B(\tau)B^T(\tau)\phi^T(t_0,\quad \tau)G^{-1}(t_0,\quad t_f)x(t_0)d\tau\\
&=\phi(t_0,\quad t_f)x(t_0)-\phi(t_0,\quad t_f)\bullet\int_{t_0}^{t_f}\phi(t_0,\quad \tau)B(\tau)B^T(\tau)\phi^T(t_0,\quad \tau)d\tau\bullet G^{-1}(t_0,\quad t_f)x(t_0)\\
&=\phi(t_0,\quad t_f)x(t_0)-\phi(t_0,\quad t_f)G(t_0,\quad t_f)G^{-1}(t_0,\quad t_f)x(t_0)\\
&=\phi(t_0,\quad t_f)x(t_0)-\phi(t_0,\quad t_f)x(t_0)\\
&=0
\end{aligned}$$

Hence, so long as Gram matrix $G(t_0,\quad t_f)$ is non-singular, time-varying system is completely controllable. The sufficiency of equation(5.28) is demonstrated.

Now let's demonstrate the necessity of equation(5.28). That's to say, if system is completely controllable, Gram matrix $G(t_0,\quad t_f)$ will be non-singular. Here, we utilize proof by contradiction to prove the necessity of equation(5.28). If Gram matrix $G(t_0,\quad t_f)$ is singular, there will have non-zero state vector $x(t_0)$ which could satisfy the following equation:

$$x^T(t_0)G(t_0,\quad t_f)x(t_0)=0$$

Namely,
$$\int_{t_0}^{t_f}x^T(t_0)\phi(t_0,\quad \tau)B(\tau)B^T(\tau)\phi^T(t_0,\quad \tau)x(t_0)d\tau=0$$

$$\int_{t_0}^{t_f}\left[B^T(\tau)\phi^T(t_0,\quad \tau)x(t_0)\right]^T\left[B^T(\tau)\phi^T(t_0,\quad \tau)x(t_0)\right]d\tau=0$$

$$\int_{t_0}^{t_f}\left\|B^T(\tau)\phi^T(t_0,\quad \tau)x(t_0)\right\|^2d\tau=0$$

But matrix $B^T(\tau)\phi^T(t_0,\quad \tau)$ is continuous for variable t, hence from above expression, we have the following expression:

$$B^T(\tau)\phi^T(t_0,\quad \tau)x(t_0)=0$$

Above state vector $x(t_0)$ is controllable for we have assumed that given system is controllable.

In terms of the definition of controllability and equation(5.27), if some non-zero state vector $x(t_0)$ is controllable, the following expression is found:

$$x(t_f)=\phi(t_0,\quad t_f)x(t_0)+\int_{t_0}^{t_f}\phi(t_f,\quad \tau)B(\tau)u(\tau)d\tau=0$$

Namely, $x(t_0)=-\phi^{-1}(t_0,\quad t_f)\int_{t_0}^{t_f}\phi(t_f,\quad \tau)B(\tau)u(\tau)d\tau=-\int_{t_0}^{t_f}\phi(t_0,\quad \tau)B(\tau)u(\tau)d\tau$

$$\|x(t_0)\|=x^T(t_0)x(t_0)=\left[-\int_{t_0}^{t_f}\phi(t_0,\quad t)B(t)u(t)dt\right]^Tx(t_0)$$

Above expression shows that state vector $x(t_0)$ is not arbitrary if it is controllable. This state vector is only such expression: $x(t_0)=0$. This conclusion is contradictory to given hypothesis.Hence the hypothesis that Gram matrix $G(t_0,\ t_f)$ is singular is not found. The necessity of equation(5.28) has been demonstrated.

Example5.7 Please judge the output controllability of the following system.

$$\dot{x}=\begin{bmatrix}0 & t\\0 & 0\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u,\quad y=\begin{bmatrix}1 & 0\end{bmatrix}x$$

Solution: System input matrix is the following:

$$B=\begin{bmatrix}0\\1\end{bmatrix}$$

(1) Ensure the state transition matrix

For the system matrix $A(t)$ satisfies the following relationship:

$$A(t_1)A(t_2)=A(t_2)A(t_1)=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

The state transition matrix $\phi(0,\ t)$ could be written into a closed form:

$$\phi(0,\ t)=E+\int_t^0 A(\tau)d\tau+\frac{1}{2!}\left[\int_t^0 A(\tau)d\tau\right]^2+\cdots=\begin{bmatrix}1 & -\frac{1}{2}t^2\\0 & 1\end{bmatrix}$$

(2) Calculate the controllability judgment matrix $G(0,\ t_f)$

$$G(0,\ t_f)=\int_{t_0}^{t_f}\phi(0,\ \tau)B(\tau)B^T(\tau)\phi^T(0,\ \tau)d\tau$$

$$=\int_{t_0}^{t_f}\begin{bmatrix}1 & -\frac{1}{2}t^2\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-\frac{1}{2}t^2 & 1\end{bmatrix}dt=\int_{t_0}^{t_f}\begin{bmatrix}\frac{1}{4}t^4 & -\frac{1}{2}t^2\\-\frac{1}{2}t^2 & 1\end{bmatrix}dt=\begin{bmatrix}\frac{1}{20}t_f^5 & -\frac{1}{6}t_f^3\\-\frac{1}{6}t_f^3 & t_f\end{bmatrix}$$

(3) Judge the singularity of controllability judgment matrix $G(0,\ t_f)$

$$\det G(0,\ t_f)=\det\begin{bmatrix}\frac{1}{20}t_f^5 & -\frac{1}{6}t_f^3\\-\frac{1}{6}t_f^3 & t_f\end{bmatrix}=\frac{1}{20}t_f^6-\frac{1}{36}t_f^6=\frac{1}{45}t_f^6$$

For $t_f>0$, $\det G(0,\ t_f)=\frac{1}{45}t_f^6>0$. Hence system is controllable on the time interval $[0,\ t_f]$.

According to above example, we can easily know that the state transition matrix of system has to be calculate firstly when the controllability of time-varying system is judged in terms of equation(5.28). But if the state transition matrix of time-varying system is not written into a closed form, above method will not be employed. In the following paragraphs, we will introduce a practical judgment criterion of system controllability. This criterion only utilizes system matrix $A(t)$ and input matrix $B(t)$ to judge system controllability.

Assume the state equation is the following:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{5.30}$$

Matrices $A(t)$ and $B(t)$ could be respectively derived for $(n-2)$ order and $(n-1)$ order, which could be marked as

$$B_1(t) = B(t)$$

$$B_i(t) = -A(t)B_{i-1}(t) + \dot{B}_{i-1}(t), \quad i = 2,3,\cdots,n$$

Order $\quad Q_c(t) = [B_1(t), \quad B_2(t), \quad \cdots, \quad B_n(t)]$

If a time t_f makes

$$rankQ_c(t) = rank[B_1(t), \quad B_2(t), \quad \cdots, \quad B_n(t)] = n \tag{5.31}$$

this system is completely controllable on the time interval $[0, \quad t_f]$.

Note that this judgment criterion is only a sufficient condition. If a system could not satisfy above condition, we would not acquire the conclusion that system is not controllable.

Example5.8 Please judge the output controllability of example5.7 by above judgment criterion.

Solution:

$$B_1 = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B_2 = -A(t)B_1 + \dot{B}_1 = -\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \end{bmatrix}$$

$$Q_c(t) = [B_1(t), \quad B_2(t)] = \begin{bmatrix} 0 & -t \\ 1 & 0 \end{bmatrix}$$

$$\det Q_c(t) = \det\begin{bmatrix} 0 & -t \\ 1 & 0 \end{bmatrix} = t$$

Obviously, if $t \neq 0$, $rankQ_c = 2 = n$; hence system is controllable on the time interval $[0, \quad t]$.

5.3 Observability of Linear System

If a system is described by the following dynamic equations

$$\dot{x} = Ax + Bu \tag{5.32a}$$

$$y = Cx + Du \tag{5.32b}$$

then

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{5.33a}$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du \tag{5.33b}$$

Since matrices A , B , C and D are known and input signal $u(t)$ is also known, the last two terms on the right-hand side of this last equation(5.33b) are known quantities. Therefore, they may be subtracted from the observed value of output $y(t)$. Hence, for investigating a sufficient and necessary condition for complete observability, it suffices to consider the system described by the following equations.

$$\dot{x} = Ax \tag{5.34a}$$

$$y = Cx \tag{5.34b}$$

5.3.1 Observability judgment criterion of time-invariant system

Now let's consider the system described by equations(5.34a) and (5.34b). The output vector $y(t)$ is

$$y = Ce^{At}x(0) \tag{5.35}$$

Referring to equation(3.35), we have

$$e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{n-1}\frac{t^k}{k!}A^k = \sum_{k=0}^{n-1}\alpha_k(t)A^k \tag{5.36}$$

Here, $\alpha_k(t) = \frac{t^k}{k!}$

Hence, we obtain

$$y = \sum_{k=0}^{n-1}C\alpha_k(t)A^kx(0)$$

or

$$y = \alpha_0(t)Cx(0) + \alpha_1(t)CAx(0) + \cdots + \alpha_{n-1}(t)CA^{n-1}x(0) \tag{5.37}$$

If the system is complete controllability, then, given the output $y(t)$ over a time interval $0 \le t \le t_1$, $x(0)$ is uniquely determined from equation(5.37). It was shown that this requires the rank of the judgment matrix to be n .

The first judgment criterion of complete state observability The sufficient and necessary condition of complete state observability of linear time-invariant system is that the rank of judgment matrix S_o is full rank or that determinant of judgment matrix S_o is non-singular, namely,

$$rankS_o = rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad (5.38)$$

Here judgment matrix S_o is called observability judgment matrix.

Example5.9 Please analyze the state controllability and state observability of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$rankS_c = rank\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 2$$

Hence, system is completely state controllable.

For output controllability, let us find the rank of the matrix $\begin{bmatrix} CB & CAB \end{bmatrix}$.

$$rankS_{ou} = rank\begin{bmatrix} CB & CAB \end{bmatrix} = rank\begin{bmatrix} 0 & 1 \end{bmatrix} = 1$$

Hence, system is completely output controllable.

For state observability, examine the rank of $\begin{bmatrix} C^T & A^T C^T \end{bmatrix}$ or $\begin{bmatrix} C \\ CA \end{bmatrix}$.

$$rankS_o = rank\begin{bmatrix} C^T & A^T C^T \end{bmatrix} = rank\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 2$$

Hence, system is completely state observable.

Example5.10 Please analyze the state observability of the following system.

$$x(k+1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{bmatrix}x(k) + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}u(k), \quad y(k) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}x(k)$$

Solution:

$$rankS_o = rank\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & -1 \\ 9 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix} = 2 < 3 = n$$

Hence, system is not observable.

Conditions for complete observability in the s **plane.** The conditions for complete observability can also be stated in terms of transfer function or transfer matrices. The necessary and sufficient conditions for complete observability is that no cancellation occurs in the transfer function or transfer matrix. If cancellation occurs, the canceled mode can not be observed in the output.

Example5.11 Please analyze the state observability of the following system.

$$\dot{x}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u,\qquad y=\begin{bmatrix}4 & 5 & 1\end{bmatrix}x$$

Note that the input signal $u(t)$ does not affect the complete observability of the system.To examine the complete state observability, we can simply set $u(t)=0$. For this system, we have

$$S_o=\begin{bmatrix}C^T & A^TC^T & (A^T)^2C^T\end{bmatrix}=\begin{bmatrix}4 & -6 & 6\\ 5 & -7 & 5\\ 1 & -1 & -1\end{bmatrix}$$

$$\det S_o=\det\begin{bmatrix}4 & -6 & 6\\ 5 & -7 & 5\\ 1 & -1 & -1\end{bmatrix}=0$$

Hence, the rank of judgment matrix S_o is less than 3. Therefore, system is not completely state observable.

In fact, in this system, cancellation occurs in the transfer function of the system.The transfer function between $X_1(s)$ and $U(s)$ is

$$\frac{X_1(s)}{U(s)}=\frac{1}{(s+1)(s+2)(s+3)}$$

And the transfer function between $Y(s)$ and $U(s)$ is

$$\frac{X_1(s)}{U(s)}=(s+1)(s+4)$$

Therefore, the transfer function between $Y(s)$ and $U(s)$ is

$$\frac{Y(s)}{U(s)}=\frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

Clearly, the two factors $(s+1)$ cancel each other. This means that there are nonzero initial states, which can not be determined from the output measurement of $y(t)$.

The transfer function has no cancellation if and only if the system is completely state controllable and completely observable. This means that the canceled transfer function does not carry along all the information characterizing the dynamic system.

5.3.2 The second criterion of complete state observability

If dynamic system is transformed into a diagonal canonical form, there will be a simpler judgment criterion of state observability. Now let us continue to discuss the complete state observability for diagonal canonical form.

Theorem The observability of linear time-constant(continuous or discrete) system always keeps unchanged by any non-singular transform.

Demonstration: We assume that the state equation of linear time-constant continuous system is as follows

$$S: \quad \dot{x} = Ax + Bu \tag{5.39}$$

Equation(5.17) after a non-singular transformation $x = P\bar{x}$ is the following:

$$\bar{S}: \quad \dot{\bar{x}} = \bar{A}x + \bar{B}u \tag{5.40}$$

In terms of observability judgment criterion, the observability matrix system S_o could be expressed as follows:

$$rankS_0 = rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.41}$$

According to the relationship $x = P\bar{x}$ between system S and system $\bar{S}$: $A = P\bar{A}P^{-1}$, $B = P\bar{B}$, $C = \bar{C}P^{-1}$, we get

$$rankS_0 = rank\begin{bmatrix} \bar{C}P^{-1} \\ \bar{C}P^{-1}P\bar{A}P^{-1} \\ \vdots \\ \bar{C}P^{-1}P\bar{A}^{n-1}P^{-1} \end{bmatrix} = rank\begin{bmatrix} \bar{C}P^{-1} \\ \bar{C}\bar{A}P^{-1} \\ \vdots \\ \bar{C}\bar{A}^{n-1}P^{-1} \end{bmatrix} = rank\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = rank\,\bar{S}_o \tag{5.42}$$

Hence, any non-singular transform can not change the observability of a system for the eigenvalues of original system matrix A and new system matrix $\bar{A}$ are same.

Complete state observability for diagonal canonical form with distinct eigenvalues If linear time-constant system which dynamic equations is $\dot{x} = Ax + Bu$, $y = Cx$ has distinct eigenvalues, the sufficient and necessary condition of complete state controllability is that elements in column of output matrix $\bar{C}$ are not all zero by non-singular transform.

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \bar{x} + \bar{B} u, \qquad y = \bar{C} x \tag{5.43}$$

For example, the observability of the following system is easily known. Here note that there are distinct eigenvalues.

(1) $\dot{x} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix} x, \quad y = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} x$; system state is completely observable.

(2) $\dot{x} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix} x, \quad y = \begin{bmatrix} 2 & 0 & 5 \end{bmatrix} x$; system state is not completely observable.

In reality, eigenvalues of system are perhaps multiple. If every multiple eigenvalue is only corresponding to a characteristic vector, system could be transformed into a Jordan canonical form, namely

$$\dot{\bar{x}} = \begin{bmatrix} J_1 & \cdots & \cdots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & J_n \end{bmatrix} \bar{x} + \bar{B} u, \qquad y = \bar{C} x \tag{5.44}$$

Complete state observability for diagonal canonical form with multiple eigenvalues If a system could be transformed into a Jordan canonical form shown in equation(5.44), the sufficient and necessary condition of complete state observability is that those columns in output matrix $\bar{C}$ which is corresponding to the first row of each Jordan block are not zero columns.

For example, let's judge the observability of the following systems.

(1) $\dot{x} = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} x, \quad y = \begin{bmatrix} 2 & 0 \end{bmatrix} x$; system state is completely observable.

(2) $\dot{x} = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} x, \quad y = \begin{bmatrix} 0 & 2 \end{bmatrix} x$; system state is not completely observable.

(3) $\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x, \quad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} x$; system state is completely observable.

5.3.3 Observability of linear time-variant system

In time-variant system, system matrix $A(t)$, input matrix $B(t)$ and output matrix $C(t)$

are both the function of time. The corresponding judgment criterion is different from that of time-constant system.Hence, it's necessary to explore the judgment criterion of time-invariant system. We should utilize the Gram matrix and non-singular transform to give out judgment criterion.

Before analyzing the observability of linear time-variant system, we firstly explore the following points:

(1) Time interval $[t_0, \quad t_f]$ is an observing time of recognizing original state of system $x(t_0)$; for time-variant system, this time interval is relative to the selection of original time t_0.

(2) If a system is not observed, the following relationship should be found:

$$C(t)\phi(t, \quad t_0)x(t_0) \equiv 0, \ t \in [t_0, \quad t_f] \tag{5.45}$$

(3) Non-linear transform of system does not change original observability of system.

Demonstration: If original state $x(t_0)$ of system is not observable, the following relationship has to be satisfied:

$$C(t)\phi(t, \quad t_0)x(t_0) \equiv 0$$

Assuming that transform matrix is sign P, we obtain:

$$x = P\bar{x}, \qquad \bar{C} = CP, \qquad C(t) = \bar{C}(t)P^{-1}$$

After above expressions are substituted into equation(5.45), we can get

$$\bar{C}(t)P^{-1}\phi(t, \quad t_0)P\bar{x}(t_0) \equiv 0$$

$$\bar{C}(t)\bar{\phi}(t, \quad t_0)\bar{x}(t_0) \equiv 0$$

Hence, original state $\bar{x}(t_0)$ is not observable. That's to say, non-linear transform does not change system observability.

(4) If original state $x(t_0)$ is not observable, new state $\alpha x(t_0)$ is not observable, too; note that coefficient α is a nonzero real.

Demonstration: If original system state $x(t_0)$ is not observed, we can get

$$C(t)\phi(t, \quad t_0)x(t_0) \equiv 0$$

Then,

$$C(t)\phi(t, \quad t_0)\alpha x(t_0) \equiv 0$$

Hence, new state $\alpha x(t_0)$ is not observed, too.

(5) If original states $x_1(t_0)$ and $x_2(t_0)$ are not observable, new state $x_1(t_0)+x_2(t_0)$ would be not observable, too.

Demonstration:If original states $x_1(t_0)$ and $x_2(t_0)$ are not observable, we could gain

$$C(t)\phi(t, \quad t_0)x_1(t_0) = C(t)\phi(t, \quad t_0)x_2(t_0) = 0$$

Then,

$$C(t)\phi(t,\quad t_0)x_1(t_0)+C(t)\phi(t,\quad t_0)x_2(t_0)=C(t)\phi(t,\quad t_0)[x_1(t_0)+x_2(t_0)]=0$$

Hence, new state $x_1(t_0)+x_2(t_0)$ is not observable,too.

A time-variant system is as follows:

$$\begin{cases}\dot{x}=A(t)x+B(t)u\\ y=C(t)x\end{cases} \tag{5.46}$$

Judgment criterion of complete output observability for time-variant system

The sufficient and necessary condition of complete state observability in time-variant system on time interval $[t_0,\quad t_f]$ is that the following Gram matrix is non-singular matrix.

$$G(t_0,\quad t_f)=\int_{t_0}^{t_f}\phi^T(t,\quad t_0)C^T(t)C(t)\phi(t,\quad t_0)dt \tag{5.47}$$

Demonstration: The solution of equation(5.46) is the following:

$$x(t)=\phi(t,\quad t_0)x(t_0)+\int_{t_0}^{t_f}\phi(t,\quad \tau)B(\tau)u(\tau)d\tau$$

Then system output is

$$y(t)=C(t)x(t)=C(t)\phi(t,\quad t_0)x(t_0)+C(t)\int_{t_0}^{t_f}\phi(t,\quad \tau)B(\tau)u(\tau)d\tau$$

When ensuring the state observability, we may neglect the role of input signal $u(t)$. And then above two equations will be converted into the following expressions:

$$x(t)=\phi(t,\quad t_0)x(t_0)$$

$$y(t)=C(t)\phi(t,\quad t_0)x(t_0)$$

Expression $\phi^T(t,\quad t_0)C^T(t)$ is multiplied into system output equation to yield

$$\phi^T(t,\quad t_0)C^T(t)y(t)=\phi^T(t,\quad t_0)C^T(t)C(t)\phi(t,\quad t_0)x(t_0)$$

Integration on time interval $[t_0,\quad t_f]$ is applied to above equation and then we gain

$$\int_{t_0}^{t_f}\phi^T(t,\quad t_0)C^T(t)y(t)dt=\int_{t_0}^{t_f}\phi^T(t,\quad t_0)C^T(t)C(t)\phi(t,\quad t_0)x(t_0)dt=G(t_0\quad t_f)x(t_0)$$

Obviously, when Gram matrix $G(t_0,\quad t_f)$ is non-singular matrix, system output $y(t)$ would uniquely ensure state $x(t_0)$.

Gram matrix could be used to judge the complete state observability of time-variant system, but its calculation work is very large. In the following, a simpler method is introduced to judge the complete state observability of time-variant system.

Assume that system matrix $A(t)$ and output matrix $C(t)$ for time variable t could be continuously differentiated respectively for $(n-2)th$ order and $(n-1)th$ order, which is marked as

$$C_1(t)=C(t)$$

$$C_i(t)=C_{i-1}(t)A(t)+\dot{C}_{i-1}(t)\quad i=2,3,\cdots,n$$

Order,

$$R(t)=\begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_n(t) \end{bmatrix} \tag{5.48}$$

On time interval $[t_0,\quad t_f]$, if $t_f > 0$ and $rankR(t)=n$, system state is observable.

Example5.12 Please judge the state observability of the following system.

$$\begin{cases} \dot{x}=\begin{bmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{bmatrix}x+B(t)u \\ y=\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}x \end{cases}$$

Solution: System matrix $A(t)$ and output matrix $C(t)$ are as follows:

$$A(t)=\begin{bmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{bmatrix}, \qquad C(t)=\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$C_1(t)=C(t)=\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$C_2(t)=C_1(t)A(t)+\dot{C}_1(t)=\begin{bmatrix} t & 1 & t^2 \end{bmatrix}$$

$$C_3(t)=C_2(t)A(t)+\dot{C}_2(t)=\begin{bmatrix} t^2+1 & 2t & t^4+2t \end{bmatrix}$$

$$R(t)=\begin{bmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \end{bmatrix}=\begin{bmatrix} 1 & 0 & 1 \\ t & 1 & t^2 \\ t^2+1 & 2t & t^4+2t \end{bmatrix}$$

Obviously, when $t>0$, $rankR(t)=3=n$; hence system state is completely observable on time interval $[0,\quad t]$.

Note that this method is only a sufficient condition. If some condition could satisfy above method, we could not draw such conclusion that system is not observable.

5.4 Diagonal Canonical Form of State Space Expression

As we have known, dynamic equations of system could be transformed into diagonal canonical form by non-singular transformation. Such mode is very helpful for analyzing and designing control system. In this section, we will discuss how to transform a system matrix into a diagonal canonical form. If readers have learned the knowledge of this section, readers may neglect the content of this section.

5.4.1 System eigenvalue and characteristic vector

If a nonzero vector υ exists in vector space, which satisfy the following equation,

$$A\upsilon = \lambda\upsilon \tag{5.49}$$

sign λ is called the eigenvalue of matrix A. Any nonzero vector υ which could satisfy equation(5.49) is called characteristic vector which is corresponding to eigenvalue λ of matrix A.

According to above definition, we try to acquire the characteristic value of matrix A. Equation(5.49) is written into equation(5.50):

$$(\lambda E - A)\upsilon = 0 \tag{5.50}$$

The sufficient and necessary condition that above non-homogeneous equation(5.50) has nonzero solution is that the determinant of expression $(\lambda E - A)$ is zero, namely:

$$\det(\lambda E - A) = |\lambda E - A| = 0 \tag{5.51}$$

Equation(5.51) is called characteristic equation. The following developed polynomial of determinant $\det(\lambda E - A)$ is called the characteristic polynomial of system matrix A.

$$\det(\lambda E - A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n \tag{5.52}$$

Roots of characteristic equation(5.52) is called eigenvalue of matrix A. According to the definition of above characteristic vector, we may know that characteristic vector is connected to eigenvalue together. If we want to calculate characteristic vector, we should firstly calculate system eigenvalues. Now let's analyze the following examples.

Example5.13 Please calculate the characteristic vector of the following matrix A.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

Solution:(1)calculate the characteristic value of matrix A.

$$\det(\lambda E - A) = \det\begin{bmatrix} \lambda & -1 & 1 \\ 6 & \lambda+11 & -6 \\ 6 & 11 & \lambda-5 \end{bmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

The solution of above equation is $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

(2)calculate the characteristic vector of eigenvalue $\lambda_1 = -1$.

We assume that characteristic vector is $\upsilon_1 = [\upsilon_{11} \quad \upsilon_{21} \quad \upsilon_{31}]^T$, and substitute λ_1 and υ_1 into equation(5.50). We could get the following equation:

$$(\lambda_1 E - A)\upsilon_1 = 0$$

Namely, $$\left(\begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}\right)\begin{bmatrix} \upsilon_{11} \\ \upsilon_{21} \\ \upsilon_{31} \end{bmatrix} = 0$$

$$\begin{bmatrix} \upsilon_{11}+\upsilon_{21}-\upsilon_{31} \\ -6\upsilon_{11}-10\upsilon_{21}+6\upsilon_{31} \\ -6\upsilon_{11}-11\upsilon_{21}+6\upsilon_{31} \end{bmatrix}=0$$

Then we can get the following equation group:

$$\begin{cases} \upsilon_{11}+\upsilon_{21}-\upsilon_{31}=0 \\ -6\upsilon_{11}-10\upsilon_{21}+6\upsilon_{31}=0 \\ -6\upsilon_{11}-11\upsilon_{21}+6\upsilon_{31}=0 \end{cases}$$

Solve above equation group and we acquire

$$\upsilon_{21}=0, \quad \upsilon_{11}=\upsilon_{31}$$

υ_{11} and υ_{31} may is any nonzero value. Here we take $\upsilon_{11}=\upsilon_{31}=1$, corresponding vector is: $\upsilon_1=\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$.

Similarly, other two vectors are also calculated which are respectively:

$$\upsilon_2=\begin{bmatrix}1 & 2 & 4\end{bmatrix}^T, \quad \upsilon_2=\begin{bmatrix}1 & 6 & 9\end{bmatrix}^T$$

5.4.2 System matrix A transformed into diagonal matrix

1.Distinct eigenvalues of matrix A

When matrix A has distinct eigenvalues, we can assume $P=\begin{bmatrix}\upsilon_1 & \upsilon_2 & \cdots & \upsilon_n\end{bmatrix}$ in which each characteristic vector υ_i has its own characteristic vector. Then there is the following relationship:

$$\bar{A}=P^{-1}AP=\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{5.53}$$

Namely, when system matrix A has distinct eigenvalues, we could transform matrix A into diagonal matrix which each element at diagonal line is respectively the eigenvalue of system matrix A by transform matrix P which is composed of characteristic vectors.

Here, we should note that the aim of constructing transform matrix P is not to calculate new system matrix $\bar{A}$, but to solve the reversible matrix P^{-1} and to calculate new matrix $\bar{B}$ and $\bar{C}$.

Example5.14 Please convert the following dynamic equations into diagonal canonical form.

$$\dot{x}=\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}u, \qquad y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}x$$

Solution: We have gotten the eigenvalues and vectors of system matrix A from example5.13.

$$\lambda_1=-1,\qquad \lambda_2=-2,\qquad \lambda_3=-3$$

$$\upsilon_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\qquad \upsilon_2=\begin{bmatrix}1\\2\\4\end{bmatrix},\qquad \upsilon_3=\begin{bmatrix}1\\6\\9\end{bmatrix}$$

Hence, transform matrix P is the following:

$$P=\begin{bmatrix}\upsilon_1 & \upsilon_2 & \upsilon_2\end{bmatrix}=\begin{bmatrix}1&1&1\\0&2&6\\1&4&9\end{bmatrix}$$

$$P^{-1}=\begin{bmatrix}3&2.5&-2\\-3&-4&3\\1&1.5&-1\end{bmatrix},\qquad \bar{A}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}$$

$$\bar{B}=P^{-1}B=\begin{bmatrix}3&2.5&-2\\-3&-4&3\\1&1.5&-1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-2\\3\\-1\end{bmatrix},\quad \bar{C}=CP=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}1&1&1\\0&2&6\\1&4&9\end{bmatrix}=\begin{bmatrix}1&1&1\end{bmatrix}$$

Then the diagonal canonical form of dynamic equation is

$$\dot{\bar{x}}=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}\bar{x}+\begin{bmatrix}-2\\3\\-1\end{bmatrix}u,\qquad y=\begin{bmatrix}1&1&1\end{bmatrix}\bar{x}$$

2.Associate matrix of distinct eigenvalues of matrix A

$$A=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_1\end{bmatrix}\tag{5.54}$$

Transform matrix P which make system matrix A be converted into diagonal matrix is a Vandermonde matrix, namely

$$P=\begin{bmatrix}1&1&1&\cdots&1\\\lambda_1&\lambda_2&\lambda_3&\cdots&\lambda_n\\\lambda_1^2&\lambda_2^2&\lambda_3^2&\cdots&\lambda_n^2\\\vdots&\vdots&\vdots&\vdots&\vdots\\\lambda_1^{n-1}&\lambda_2^{n-1}&\lambda_3^{n-1}&\cdots&\lambda_n^{n-1}\end{bmatrix}\tag{5.55}$$

In equation(5.55), λ_1, $\lambda_2,\cdots$, λ_n are distinct eigenvalues of matrix A.

Example5.15 Please convert the following system matrix into diagonal matrix.

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}$$

Solution: (1) calculate the eigenvalues of system matrix A.

$$\det(\lambda E-A)=\det\begin{bmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ 6 & 11 & \lambda+6\end{bmatrix}=\lambda^3+6\lambda^2+11\lambda+6=0$$

We solve above equation to get corresponding solution: $\lambda_1=-1$, $\lambda_2=-2$, $\lambda_3=-3$.

Hence, transform matrix is the following:

$$P=\begin{bmatrix}1 & 1 & 1\\ -1 & -2 & -3\\ 1 & 4 & 9\end{bmatrix}$$

Diagonal matrix is the following:

$$\bar{A}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}$$

3.Matrix A has multiple eigenvalues but n independent characteristic vectors

For this case, we take $P=[\upsilon_1 \quad \upsilon_2 \quad \cdots \quad \upsilon_n]$ to make system matrix A become a diagonal matrix.

Example5.16 Please convert the following system matrix into diagonal matrix.

$$A=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Solution: Characteristic equation is the following:

$$\det(\lambda E-A)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & 0 & 1\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda-3\end{vmatrix}=(\lambda-1)^2(\lambda-2)=0$$

Characteristic roots are respectively: $\lambda_1=\lambda_2=2$, $\lambda_3=2$.

Acquire the characteristic vector of eigenvalue λ_1 or λ_2.

$$(\lambda_1 E-A)\upsilon_1=0$$

Substitute eigenvalue λ_1 into above equation:

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\upsilon_{11}\\ \upsilon_{21}\\ \upsilon_{31}\end{bmatrix}=0$$

Solution is $\upsilon_{31}=0$; obviously υ_{21} and υ_{11} are not constrained, which are any nonzero value. If υ_{21} and υ_{11} are independent value, we think that they are the solution of above equation. Here two independent vectors are taken as follows:

$$\upsilon_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \upsilon_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Obviously, these two vectors are independent.

Solve the characteristic vector of eigenvalue λ_3.

$$(\lambda_3 E - A)\upsilon_3 = 0, \begin{bmatrix} \lambda_3 - 1 & 0 & 1 \\ 0 & \lambda_3 - 1 & 0 \\ 0 & 0 & \lambda_3 - 2 \end{bmatrix} \begin{bmatrix} \upsilon_{13} \\ \upsilon_{23} \\ \upsilon_{33} \end{bmatrix} = 0$$

Namely, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon_{13} \\ \upsilon_{23} \\ \upsilon_{33} \end{bmatrix} = 0$

After simplifying above equation group, we could get :

$$\begin{cases} \upsilon_{13} + \upsilon_{33} = 0 \\ \upsilon_{23} = 0 \end{cases}$$

Order $\upsilon_{13} = -1$, we can get characteristic vector: $\upsilon_3 = [\upsilon_{13} \quad \upsilon_{23} \quad \upsilon_{33}]^T = [-1 \quad 0 \quad 1]^T$.

Hence, transfer function is such:

$$P = [\upsilon_1 \quad \upsilon_2 \quad \upsilon_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We easily testify the diagonal system matrix $\bar{A}$:

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

5.4.3 System matrix A transformed into Jordan matrix

If system matrix A is not transformed into diagonal matrix, it has to been transformed into Jordan matrix which is similar to diagonal matrix.

1. The definition of Jordan block and Jordan matrix

Matrix form is as follows:

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \tag{5.56}$$

Matrix block as equation(5.56) is called Jordan block. Quasi diagonal matrix which is composed of a lot of Jordan blocks is called Jordan matrix. In Jordan matrix, every Jordan block which is located in the diagonal of matrix is called Jordan block. The Jordan matrix is the following:

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

In above matrix, eigenvalue λ_1 which has five characteristic vectors constitutes two Jordan blocks; eigenvalue λ_2 constitute a Jordan block.

For common matrix, even if eigenvalues have been solved, it is very difficult to gain the corresponding transfer function P except some special cases.

2. General characteristic vector

If system matrix A has some multiple characteristic roots, the number of independent state vectors which is ensured by equation $(\lambda E - A)\upsilon = 0$ is less than the dimension of system matrix A . To form a transfer matrix P , we need to confirm some characteristic vectors which is called general character vector.

(1)Definition of general character vector

When $(\lambda E - A)^k \upsilon = 0$ and $(\lambda E - A)^{k-1} \upsilon \neq 0$, vector υ is called general characteristic vector of system matrix A .

(2)General character vector group

We assume that sign υ is general characteristic vector which is produced by system matrix A which eigenvalue and rank are respectively λ and k . And the following expression group is always found:

$$\begin{aligned} \upsilon_k &= \upsilon \\ \upsilon_{k-1} &= (A - \lambda E)\upsilon = (A - \lambda E)\upsilon_k \\ \upsilon_{k-2} &= (A - \lambda E)^2 \upsilon = (A - \lambda E)\upsilon_{k-1} \\ &\vdots \\ \upsilon_1 &= (A - \lambda E)^{k-1} \upsilon = (A - \lambda E)\upsilon_2 \end{aligned} \tag{5.57}$$

The row vector $[\upsilon_1 \quad \upsilon_2 \quad \cdots \quad \upsilon_k]$ is called the general eigenvector group of eigenvalue λ .

(3)Transform matrix P

Transform matrix P which converts system matrix A into Jordan matrix is composed of the general characteristic vector group of each eigenvalue, namely:

$$P = \begin{bmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_n \end{bmatrix} \tag{5.58}$$

Here, give an example to illustrate above theories.

Example5.17 Please convert the following system matrix into Jordan matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: Characteristic equation is the following:

$$\det(\lambda E - A) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 1 & -3 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 1)(\lambda - 2) = 0$$

The solution solved is: $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$.

$$(A - \lambda_1 E) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad rank(A - \lambda_1 E) = 2$$

Since the rank of expression $(A - \lambda_1 E)$ is 2, there is only one independent characteristic vector of eigenvalue $\lambda_1 = \lambda_2 = 1$. General eigenvalue has to be acquired and constructed according to equation(5.57). In this example, nonzero vector υ needs to satisfy the following equations.

$$(A - \lambda_1 E)\upsilon = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_{11} \\ \upsilon_{21} \\ \upsilon_{31} \end{bmatrix} = \begin{bmatrix} \upsilon_{21} + 2\upsilon_{31} \\ 3\upsilon_{31} \\ \upsilon_{31} \end{bmatrix} \neq 0$$

$$(A - \lambda_1 E)^2 \upsilon = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \upsilon_{11} \\ \upsilon_{21} \\ \upsilon_{31} \end{bmatrix} = \begin{bmatrix} 7\upsilon_{31} \\ 6\upsilon_{31} \\ 4\upsilon_{31} \end{bmatrix} = 0$$

After solving expression $(A - \lambda_1 E)^2 \upsilon = 0$, we can gain $\upsilon_{31} = 0$. Hence, we can acquire

$$(A - \lambda_1 E)\upsilon = \begin{bmatrix} \upsilon_{21} \\ 0 \\ 0 \end{bmatrix} \neq 0$$

Then $\upsilon_{21} \neq 0$, and we should note that υ_{11} could get any value. We get the following values for variables:

$$\upsilon_{21} = 1, \; \upsilon_{31} = 0, \; \upsilon_{11} = 0$$

According to equation(5.57), we can get the general characteristic vector of eigenvalue $\lambda_1 = \lambda_2 = 1$.

$$\upsilon = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \qquad \upsilon_2 = \upsilon = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

The second characteristic vector υ_1 is the following:

$$\upsilon_1 = (A-\lambda_1 E)\upsilon_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the general characteristic vector group is expressed as follows:

$$[\upsilon_1 \quad \upsilon_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Calculate the eigenvector $\upsilon_3 = [\upsilon_{13} \quad \upsilon_{32} \quad \upsilon_{33}]^T$ of eigenvalue $\lambda_3 = 2$.

$$(A-\lambda_3 E)\upsilon_3 = 0$$

Namely,

$$\left\{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right\}\begin{bmatrix} \upsilon_{13} \\ \upsilon_{23} \\ \upsilon_{33} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \upsilon_{13} \\ \upsilon_{23} \\ \upsilon_{33} \end{bmatrix} = 0$$

Above matrix equation could be written into the following equation group:

$$\begin{cases} -\upsilon_{13} + \upsilon_{23} + 2\upsilon_{33} = 0 \\ -\upsilon_{23} + 3\upsilon_{33} = 0 \end{cases}$$

Solve this equation group and gain the following expressions:

$$\upsilon_{23} = 3\upsilon_{33}, \quad \upsilon_{13} = 5\upsilon_{33}$$

Take $\upsilon_{33} = 1$ and then get corresponding vector: $\upsilon_3 = [\upsilon_{13} \quad \upsilon_{32} \quad \upsilon_{33}]^T = [5 \quad 3 \quad 1]^T$.

Hence, transform matrix P which can transform system matrix A into Jordan matrix is the following:

$$P = [\upsilon_1 \quad \upsilon_2 \quad \upsilon_3] = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Jordan matrix of system matrix A is the following:

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This Jordan matrix includes two Jordan blocks.

5.4.4 Pattern matrix

When the eigenvalue of matrix is complex, above method is suitable; but the element at diagonal will be complex. In order to avoid that the matrix which includes complex appears, we need to utilize pattern matrix to express such complex case.

If a matrix has m eigenvalues and l complex eigenvalue groups, linear transform matrix P could transform system matrix A into the following form:

$$\bar{A}=\begin{bmatrix}\lambda_1 & & & & & & & 0\\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_m & & & & \\ & & & & M_1 & & & \\ & & & & & M_2 & & \\ & & & & & & \ddots & \\ 0 & & & & & & & M_l\end{bmatrix} \tag{5.59}$$

In expression(5.59), $M_i=\begin{bmatrix}\sigma_i & \omega_i\\ -\omega_i & \sigma_i\end{bmatrix}$, $(i=1,2,3,\cdots,l)$; matrix $\bar{A}$ in equation(5.59) is called pattern matrix.

If the number of j-th complex eigenvalues is r, and if the number of corresponding characteristic vector is one, complex characteristic value M_j could be written into the following form:

$$M_j=\begin{bmatrix}T_j & E & 0 & \cdots & 0\\ 0 & T_j & E & & 0\\ 0 & 0 & T_j & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & E\\ 0 & 0 & 0 & \cdots & T_j\end{bmatrix}_{2r\times 2r} \tag{5.60}$$

In expression(5.60), $T_j=\begin{bmatrix}\sigma_j & \omega_j\\ -\omega_j & \sigma_j\end{bmatrix}$, $E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

If matrix A has multiple characteristic values, the corresponding parts in matrix $\bar{A}$ can be written into Jordan blocks. Pattern matrix is not diagonal matrix, but its elements is composed of the components of eigenvalue. Then transform matrix P will become real matrix. Hence, pattern matrix could avoid the calculation of complex matrix.

Example5.18 Please convert the following system matrix into pattern matrix.

$$A=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}$$

Solution: Step1, calculate the characteristic roots of characteristic equation.

$$|\lambda E-A|=\begin{vmatrix}\lambda & -1\\ 2 & \lambda+2\end{vmatrix}=\lambda^2+2\lambda+2=0$$

Characteristic roots are: $\lambda_1=-1+j$, $\lambda_2=-1-j$.

Step2, acquire the characteristic vector υ_1 of eigenvalue λ_1.

$$(A-\lambda_1 E)\upsilon_1=\begin{bmatrix}1-j & 1\\ -2 & -1-j\end{bmatrix}\begin{bmatrix}\upsilon_{11}\\ \upsilon_{21}\end{bmatrix}=\begin{bmatrix}\upsilon_{11}+\upsilon_{21}-j\upsilon_{11}\\ -2\upsilon_{11}-\upsilon_{21}-j\upsilon_{21}\end{bmatrix}=0$$

Here, we assume $\upsilon_{11} = a_1 + jb_1$, $\upsilon_{21} = a_2 + jb_2$; and then there is the following equation group:

$$\begin{cases} a_1 + jb_1 + a_2 + jb_2 - ja_1 + b_1 = 0 \\ -2a_1 - j2b_1 - a_2 - jb_2 - ja_2 + b_2 = 0 \end{cases}$$

Arrange above equation group and then gain

$$\begin{cases} (a_1 + a_2 + b_1) + j(b_1 + b_2 - a_1) = 0 \\ (b_2 - 2a_1 - a_2) - j(2b_1 + b_2 + a_2) = 0 \end{cases}$$

The condition that complex is zero is that real and imaginary parts are zero; hence we have:

$$\begin{cases} a_1 + a_2 + b_1 = 0 \\ b_1 + b_2 - a_1 = 0 \\ b_2 - 2a_1 - a_2 = 0 \\ 2b_1 + b_2 + a_2 = 0 \end{cases}$$

Eliminate redundant equations and then gain

$$\begin{cases} b_1 = -a_1 - a_2 \\ b_2 = 2a_1 + a_2 \end{cases}$$

Order $a_1 = 1$, $a_2 = -1$ and then we can know $b_1 = 0$ and $b_2 = 1$. Hence,

$$\upsilon_1 = \begin{bmatrix} 1 \\ -1+j \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + j\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha_1 + j\beta_1,\ \alpha_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \beta_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Transform matrix P is

$$P = [\alpha_1 \quad \beta_1] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Pattern matrix is

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Generally, if the eigenvalues of a 2×2 matrix A are $\lambda_1 = \sigma + j\omega$ and $\lambda_2 = \sigma - j\omega$, matrix A could be transformed into pattern matrix $\bar{A}$ which transform matrix is $P = [\alpha_1 \quad \beta_1]$. Here, sign α_1 and sign β_1 are respectively real and imaginary of complex, namely, $\upsilon_1 = \alpha_1 + j\beta_1$.

For $n \times n$ matrix, if there are $n-2$ different eigenvalues $\lambda_1 \ \cdots \ \lambda_{n-2}$ and a pair of complex eigenvalue $\sigma \pm j\omega$, such matrix could be transformed and simplified into the following pattern matrix:

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda_2 & & \vdots & \vdots & 0 \\ 0 & & \ddots & \vdots & 0 & \vdots \\ \vdots & & & \lambda_{n-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sigma & \omega \\ 0 & 0 & \cdots & 0 & -\omega & \sigma \end{bmatrix} \tag{5.61}$$

Its transform matrix P is

$$P = \begin{bmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_{n-2} & \alpha_1 & \beta_1 \end{bmatrix} \tag{5.62}$$

In equation(5.62), υ_i $(i = 1,2,3,\cdots,n-2)$ is the corresponding vector of distinct eigenvalue λ_i . Sign α_1 and sign β_1 are respectively real and imaginary of conjugate complex characteristic vector.

5.5 Dual relation between controllability and observability

There exists in internal relationship between controllability and observability. Such internal relation could be confirmed by dual principle which is firstly put forward by Kalman. That's to say, a controllability of system is equal to the observability of its dual system.

5.5.1 Dual relation of linear system

We assume that there are two systems. One system $\sum_1$ is

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 , \qquad y_1 = C_1 x_1$$

Another system $\sum_2$ is

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 , \qquad y_2 = C_2 x_2$$

If the following conditions could be satisfied, we think that system $\sum_1$ and system $\sum_2$ are dual.

$$A_2 = A_1^T , \quad B_2 = C_1^T , \ C_2 = B_1^T \tag{5.63}$$

In equation(5.63), vectors x_1 and x_2 is state vectors with n dimension; input control signals u_1 and u_2 are respectively r -dimension and m -dimension control vectors; output control signals y_1 and y_2 are respectively m -dimension and r -dimension output vectors; matrix A_1 , A_2 —system matrix; control matrix B_1 , B_2 — $n \times r$ and $n \times m$ control matrix; matrix C_1 , C_2 — $m \times n$ and $r \times n$ output matrix.

Obviously, system $\sum_1$ is a r -input and m -output and n -order system, its dual system $\sum_2$ is a m -input, r -output and n -order system. The block diagram of dual systems $\sum_1$ and $\sum_2$ is shown in figure5.3.

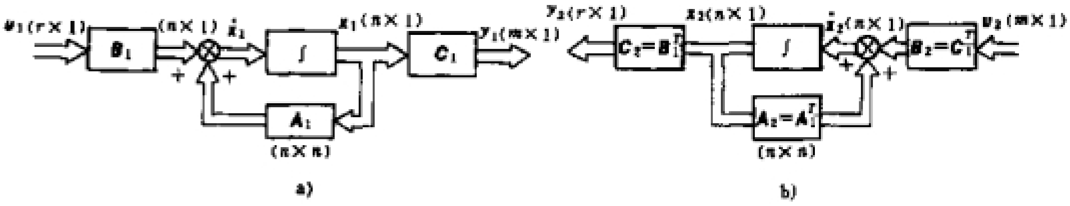

Figure5.3 Simulating structural diagram of dual systems

From the block diagram of figure5.3, we obviously know that inputs and outputs exchange in dual systems and that the direction of signal delivery is opposite and that corresponding matrix is transposed and that comparing point and outlet point are exchanged.

Observing above block diagram from the viewpoint of transfer matrix, we can get the transfer function $G_1(s)$ in figure5.3(a) which is a $m \times r$ matrix.

$$G_1(s) = C_1(sE - A_1)^{-1} B_1 \tag{5.64}$$

The transfer function $G_2(s)$ in figure5.3(b) is a $r \times m$ matrix which is as follows:

$$G_2(s) = C_2(sE - A_2)^{-1} B_2 = B_1^T(sE - A_1^T)^{-1} C_1^T = B_1^T\left[(sE - A_1)^{-1}\right]^T C_1^T \tag{5.65}$$

Transpose the transfer matrix $G_2(s)$ and then gain

$$[G_2(s)]^T = \left[B_1^T\left[(sE - A_1)^{-1}\right]^T C_1^T\right]^T = C_1(sE - A_1)^{-1} B_1 \tag{5.66}$$

Obviously, the transfer matrices of dual systems transpose each other. Similarly, the input-state transfer function of a system and state-output transfer function of dual system transpose each other. However, state-output transfer function of original system and input-state transfer function of dual system transpose each other, too.

Of course, we need to point out that the characteristic equations of each other dual systems are also same, namely

$$|sE - A_1| = |sE - A_2| \tag{5.67}$$

for the following expression is always found.

$$|sE - A_1| = |sE - A_2^T| = |sE - A_2| \tag{5.68}$$

5.5.2 Dual principle

If system $\sum_1 = (A_1,\ B_1,\ C_1)$ and system $\sum_2 = (A_2,\ B_2,\ C_2)$ are each other dual systems, the controllability of system $\sum_1$ is equal to the observability of system $\sum_2$, and the observability of system $\sum_1$ is equal to the controllability of system $\sum_2$. Or, that's to say, if the system $\sum_1$ is completely controllable(observable), the dual system $\sum_2$

will be completely observable (controllable).

Demonstration:

For system $\sum_1$, judgment matrix of system controllability is the following:

$$S_{c2}=\begin{bmatrix}B_2 & A_2B_2 & A_2^2B_2 & \cdots & A_2^{n-1}B_2\end{bmatrix}$$

If $rankS_{c2}=n$, the state of system $\sum_1$ is completely controllable.

The relationship in equations(5.63) is substituted into above judgment matrix S_{c2} and then we get

$$S_{c2}=\begin{bmatrix}C_1^T & A_1^TC_1^T & \left(A_1^T\right)^2C_1^T & \cdots & \left(A_1^T\right)^{n-2}C_1^T\end{bmatrix}=S_{o1}^T$$

This expression demonstrates that the rank of observable judgment matrix S_{o1} of system $\sum_1$ is n, and then it also shows that system $\sum_1$ is completely observable.

Similarly, $S_{o2}^T=\begin{bmatrix}C_2^T & A_2^TC_2^T & \cdots & \left(A_2^T\right)^{n-1}C_2^T\end{bmatrix}=\begin{bmatrix}B_1 & A_1B_1 & \cdots & A_1^{n-1}B_1\end{bmatrix}=S_{c1}$

Namely, if the rank of observable judgment matrix S_{o2} is full rank, system $\sum_2$ is completely observable; then the rank of controllable judgment matrix S_{c1} is full rank, system $\sum_1$ is completely controllable.

5.5.3 Dual principle of time-variant system

For time-variant systems $\sum_1=[A_1(t),\ B_1(t),\ C_1(t)]$ and $\sum_2=[A_2(t),\ B_2(t),\ C_2(t)]$, if the following relationship is found, system $\sum_1$ and system $\sum_2$ are each other duality.

$$\begin{cases}A_2(t)=-A_1^T(t)\\ B_2(t)=C_1^T(t)\\ C_2(t)=B_1^T(t)\end{cases}\tag{5.69}$$

According to above definition, we may deduce that their state transfer matrices transpose each other, namely

$$\phi_2^T(t_0,\ t)=\phi_1(t,\ t_0)\tag{5.70}$$

In equation(5.70), $\phi_1(t,\ t_0)$ is the state transfer matrix of system $\sum_1$, $\phi_2(t,\ t_0)$ is the state transfer matrix of system $\sum_2$.

Demonstration:

For system $\sum_2$, we can gain the following state equation:

$$\dot{x}_2=A_2(t)x_2+B_2(t)u_2(t)=-A_1^T(t)x_2+C_1^T(t)u_2(t)\tag{5.71}$$

Its state transfer matrix should satisfy the following differential equation:

$$\begin{cases} \dot{\phi}_2(t, \quad t_0) = -A_1^T(t)\phi_2(t, \quad t_0) \\ \phi_2(t_0, \quad t_0) = E \end{cases} \tag{5.72}$$

Transpose equation $\phi_2(t_0, \quad t_0) = E$ and we acquire

$$\phi_2^T(t_0, \quad t_0) = E, \qquad \phi_2^T(t_0, \quad t)\phi_2^T(t, \quad t_0) = E$$

Differentiate above matrix equation and yield

$$\frac{d}{dt}\left[\phi_2^T(t_0, \quad t)\phi_2^T(t, \quad t_0)\right] = 0$$

$$\dot{\phi}_2^T(t_0, \quad t)\phi_2^T(t, \quad t_0) + \phi_2^T(t_0, \quad t)\dot{\phi}_2^T(t, \quad t_0) = 0$$

Namely,

$$\dot{\phi}_2^T(t_0, \quad t) = -\phi_2^T(t_0, \quad t)\dot{\phi}_2^T(t, \quad t_0)\phi_2^T(t_0, \quad t) \tag{5.73}$$

Transpose equation(5.72) and we can know

$$\dot{\phi}_2^T(t, \quad t_0) = -\phi_2^T(t, \quad t_0)A_1(t)$$

Above expression is substituted into equation(5.73) and we can gain:

$$\dot{\phi}_2^T(t_0, \quad t) = \phi_2^T(t_0, \quad t)\phi_2^T(t, \quad t_0)A_1(t)\phi_2^T(t_0, \quad t) = A_1(t)\phi_2^T(t_0, \quad t)$$

According to the property of state transition matrix, $\phi_2^T(t_0, \quad t)$ has to be the state transition matrix of the following system.

$$\dot{x}_1 = A_1(t)x_1$$

Namely, $\phi_1(t_0, \quad t) = \phi_2^T(t_0, \quad t)$

That's to say, system $\sum_1$ and system $\sum_2$ are each other dual systems which state transition matrices are each other transposition matrix. Hence we can gain the dual principle of time-variant system: the observability of system $\sum_1$ is equivalent to the controllability of system $\sum_2$, the controllability of system $\sum_1$ is equivalent to the observability of system $\sum_2$.

5.6 Controllability and Observability Standardization of State Space Expression

As we have known, for a same system there may be different state space expressions, some specific form is called canonical form.It is very helpful for control system design to standardize the state space expression. For example, diagonal form is conveniently utilized to judge the controllability and observablity of system, but it is not perhaps suitable for state feedback design. In this section, controllable canonical form and observable canonical form will be introduced to help readers design state feedback components or state observer.

Let's firstly discuss the linear continuous and discrete time-invariant systems which are marked as $\{A, \quad b, \quad c\}$. The system by linear transform is marked as $\left\{\bar{A}, \quad \bar{b}, \quad \bar{c}\right\}$. Non-singular transform will not change the controllability and observability of system.

For linear time-invariant continuous system with one input, the state space expression of system could be expressed as the following:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = cx(t) \end{cases} \tag{5.75}$$

System transfer function is

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{5.76}$$

Linear transform is

$$x(t) = P\bar{x}(t) \quad \text{or} \quad \bar{x}(t) = Tx(t) \tag{5.77}$$

The state equation after linear transform is

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x}(t) + \bar{b}u(t) \\ y(t) = \bar{c}\bar{x}(t) \end{cases} \tag{5.78}$$

For discrete system, the state space expression is

$$\begin{cases} \dot{x}(k+1) = Ax(k) + bu(k) \\ y(k) = cx(k) \end{cases} \tag{5.79}$$

Linear transform is

$$x(k) = P\bar{x}(k) \quad \text{or} \quad \bar{x}(k) = Tx(k) \tag{5.80}$$

The state equation after linear transform is

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{b}u(k) \\ y(k) = \bar{c}\bar{x}(k) \end{cases} \tag{5.81}$$

5.6.1 First controllable canonical form

Definition: the state equation with the following form is called first controllable canonical form.

$$\bar{A}=\begin{bmatrix}0 & 1 & & & 0\\ \vdots & \vdots & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \cdots & 1\\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1}\end{bmatrix},\quad \bar{b}=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix} \tag{5.82}$$

The controllable canonical form is limited to the form of matrix A and column vector b. Sign c can be arbitrary.

Theorem The state equation with first controllable canonical form is controllable. Any a controllable state equation can be transformed into first controllable canonical form by non-singular form $\bar{x}=Tx$.

$$T=\begin{bmatrix}T_1\\ T_1A\\ T_1A^2\\ \vdots\\ T_1A^{n-1}\end{bmatrix},\quad T_1=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}S_c^{-1},\quad S_c=\begin{bmatrix}b & Ab & \cdots & A^{n-1}b\end{bmatrix} \tag{5.83}$$

Example5.19 Please transform the following state equation into first controllable canonical form.

$$\dot{x}=\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}x+\begin{bmatrix}-1\\ 1\end{bmatrix}u$$

Solution: Step1, check the controllability of system.

$$.S_c=\begin{bmatrix}-1 & -1\\ 1 & 3\end{bmatrix},\quad rankS_c=2$$

According to judgment criterion of controllability, system state is controllable, hence given system could be transformed into first controllable canonical form. System characteristic equation is

$$\det(sE-A)=\begin{vmatrix}s-1 & 0\\ 1 & s-2\end{vmatrix}=s^2-3s+2$$

According to equation(5.82), first controllable canonical form is the following:

$$\bar{A}=\begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix},\qquad \bar{b}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Step2, acquire transform matrix T.

$$S_c^{-1}=\begin{bmatrix}-3/2 & -1/2\\ 1/2 & 1/2\end{bmatrix},\qquad T_1=\begin{bmatrix}0 & 1\end{bmatrix}S_c^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

Hence, transform matrix T is the following:

$$T=\begin{bmatrix}T_1\\ T_1A\end{bmatrix}=\begin{bmatrix}1/2 & 1/2\\ 0 & 1\end{bmatrix}$$

Step3, validate canonical form.

$$T^{-1}=\begin{bmatrix}2 & -1\\ 0 & 1\end{bmatrix},\qquad \bar{A}=TAT^{-1}=\begin{bmatrix}1/2 & 1/2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix}$$

$$\bar{b}=Tb=\begin{bmatrix}1/2 & 1/2\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Example5.20 Please transform the following dynamic equations into first controllable canonical form.

$$\dot{x}=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 1\\ 0 & 2 & 0\end{bmatrix}x+\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}u,\qquad y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

Solution:

Step1, check system controllability.

$$S_c=\begin{bmatrix}2 & 4 & 16\\ 1 & 6 & 8\\ 1 & 2 & 12\end{bmatrix},\qquad S_c^{-1}=\begin{bmatrix}1.75 & -0.5 & -2\\ -0.125 & 0.25 & 0\\ -0.125 & 0 & 0.25\end{bmatrix}$$

$DetS_c\neq 0$, $rankS_c=3$; hence given system is controllable which could be transformed into controllable canonical form. System characteristic equation is the following:

$$\det(sE-A)=\begin{vmatrix}s-1 & -2 & 0\\ -3 & s+1 & -1\\ 0 & -2 & s\end{vmatrix}=s^3-9s+2$$

The first controllable canonical form is such:

$$\bar{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 9 & 0\end{bmatrix},\qquad \bar{b}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

Step2, acquire transform matrix T

$$T_1=\begin{bmatrix}0 & 0 & 1\end{bmatrix}S_c^{-1}=\begin{bmatrix}-0.125 & 0 & 0.25\end{bmatrix},\quad T_1A=\begin{bmatrix}-0.125 & 0.25 & 0\end{bmatrix}$$

$$T_1A^2=\begin{bmatrix}-0.125 & 0.25 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 1\\ 0 & 2 & 0\end{bmatrix}=\begin{bmatrix}0.625 & -0.5 & 0.25\end{bmatrix}$$

Transform matrix T is the following:

$$T=\begin{bmatrix}T_1\\ T_1A\\ T_1A^2\end{bmatrix}=\begin{bmatrix}-0.125 & 0 & 0.25\\ -0.125 & 0.25 & 0\\ 0.625 & -0.5 & 0.25\end{bmatrix}$$

Step3, check the validity of controllable canonical form.

$$T^{-1}=\begin{bmatrix}-2 & 4 & 2\\ -1 & 6 & 1\\ 3 & 2 & 1\end{bmatrix},\qquad \bar{b}=Tb=\begin{bmatrix}-0.125 & 0 & 0.25\\ -0.125 & 0.25 & 0\\ 0.625 & -0.5 & 0.25\end{bmatrix}\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$\bar{A}=TAT^{-1}=\begin{bmatrix}-0.125 & 0 & 0.25\\ -0.125 & 0.25 & 0\\ 0.625 & -0.5 & 0.25\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 1\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}-2 & 4 & 2\\ -1 & 6 & 1\\ 3 & 2 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 9 & 0\end{bmatrix}$$

$$\bar{c}=cT^{-1}=\begin{bmatrix}3 & 2 & 1\end{bmatrix}$$

5.6.2 Second controllable canonical form

Definition: State equation with the following form is called the second controllable canonical form.

$$\bar{A}=\begin{bmatrix}0 & \cdots & 0 & -a_0\\ 1 & 0 & \vdots & \vdots\\ \vdots & \ddots & \ddots & -a_{n-2}\\ 0 & \cdots & 1 & -a_{n-1}\end{bmatrix},\qquad \bar{b}=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix} \tag{5.84}$$

Theorem: The state equation with the second controllable canonical form is controllable. Any a controllable state equation can be transformed into the second controllable canonical form which linear transform is $x=P\bar{x}$. Here transform matrix P is the following:

$$P=S_c=\begin{bmatrix}b & Ab & \cdots & A^{n-1}b\end{bmatrix} \tag{5.85}$$

Example5.21 Please transform system in example5.20 into second controllable canonical form.

Solution:

In terms of the result of example5.20, we can know the judgment matrix of controllability which is also transform matrix of the second controllable canonical form.

$$P=S_c=\begin{bmatrix}2 & 4 & 16\\ 1 & 6 & 8\\ 1 & 2 & 12\end{bmatrix},\qquad P^{-1}=S_c^{-1}=\begin{bmatrix}1.75 & -0.5 & -2\\ -0.125 & 0.25 & 0\\ -0.125 & 0 & 0.25\end{bmatrix}$$

Testify the correctness of transform matrix P.

$$\bar{A}=P^{-1}AP=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 9\\ 0 & 1 & 0\end{bmatrix},\qquad \bar{b}=P^{-1}b=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$\bar{c}=cP=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 16\\ 1 & 6 & 8\\ 1 & 2 & 12\end{bmatrix}=\begin{bmatrix}1 & 2 & 12\end{bmatrix}$$

5.6.3 First observable canonical form

Definition: State space expression with the following form is called the first observable canonical form.

$$\bar{A}=\begin{bmatrix}0 & \cdots & 0 & -a_0\\ 1 & 0 & \vdots & \vdots\\ \vdots & \ddots & \ddots & -a_{n-2}\\ 0 & \cdots & 1 & -a_{n-1}\end{bmatrix},\qquad \bar{c}=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix} \tag{5.85}$$

Note that the observable canonical form is only limited to given form of matrices $\bar{A}$, $\bar{c}$; matrix $\bar{b}$ may be arbitrary.

Theorem Dynamic equation with the first observable canonical form is observable. Any an observable dynamic equation is both transformed into the first observable canonical form which transform relation is $x=P\bar{x}$. Transform matrix P can be ensured through the following expressions.

$$P=\begin{bmatrix}p_1 & Ap_1 & A^2p_1 & \cdots & A^{n-1}p_1\end{bmatrix} \tag{5.86a}$$

$$p_1=\begin{bmatrix}c\\ cA\\ \vdots\\ cA^{n-1}\end{bmatrix}^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}=S_o^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix} \tag{5.86b}$$

Example5.22 Please transform the following dynamic equations into first observable canonical form.

$$\dot{x}=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & 1\\ 0 & 2 & 0\end{bmatrix}x+\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}u,\qquad y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

Solution:

Step1, check given system observability.

$$S_o=\begin{bmatrix}c\\ cA\\ cA^2\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 2 & 0\\ 6 & -2 & 2\end{bmatrix},\quad \det S_o\neq 0,\ \ rankS_o=3$$

Hence, given system is observable and it can be transformed into first observable canonical form. System characteristic equation is such:

$$\det(sE-A)=\begin{vmatrix}s-1 & -2 & 0\\ -3 & s+1 & -1\\ 0 & -2 & s\end{vmatrix}=s^3-9s+2=0$$

In terms of equation(5.85), we can get the first observable canonical form:

$$\bar{A}=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 9\\ 0 & 1 & 0\end{bmatrix},\qquad \bar{c}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

Step2, gain the transform matrix P.

$$S_o^{-1}=\begin{bmatrix}-1/3 & 1/6 & 1/6\\ 0 & 1/2 & 0\\ 1 & 0 & 0\end{bmatrix}$$

In terms of equation(5.86), we can gain the transform matrix P.

$$p_1=S_o^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1/6\\0\\0\end{bmatrix},\quad Ap_1=\begin{bmatrix}1/6\\1/2\\0\end{bmatrix},\qquad A^2p_1=\begin{bmatrix}7/6\\0\\1\end{bmatrix}$$

$$P=\begin{bmatrix}p_1 & Ap_1 & A^2p_1\end{bmatrix}=\begin{bmatrix}1/6 & 1/6 & 7/6\\ 0 & 1/2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

5.6.4 Second observable canonical form

Definition: State space expression with the following form is called the second observable canonical form.

$$\bar{A}=\begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & 1\\ -a_0 & -a_1 & \cdots & -a_{n-1}\end{bmatrix},\qquad \bar{c}=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \tag{5.87}$$

Theorem Dynamic equation with the second observable canonical form is observable. Any an observable dynamic equation is both transformed into the second observable canonical form which transform relation is $\bar{x}=Tx$. Transform matrix T can be ensured through the following expressions.

$$T=S_o=\begin{bmatrix}c\\ cA\\ \vdots\\ cA^{n-1}\end{bmatrix} \tag{5.88}$$

5.7 Canonical Decomposition of Linear System

If a system is not completely controllable and observable, this system includes uncontrollable and unobservable subsystems. As we know, non-singular linear transform does not change the controllability and observability of given system, hence system state equation can be divided into controllable and observable, controllable and unobservable, uncontrollable and observable, uncontrollable and unobservable subsystems.Then system structure characteristics can be further discovered.

5.7.1 Canonical decomposition of controllability

Assume that the dynamic equation of linear time-invariant system is as follows:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.89}$$

If equation(5.89) is completely controllable, namely the rank of controllable judgment matrix S_c is $n_1 < n$, a non-singular transform relation $x = P_c \bar{x}$ makes given system become the following form:

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \tag{5.90a}$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} \tag{5.90b}$$

Among equation(5.90), n_1 -dimension subsystem is controllable, which could be expressed as follows:

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u \tag{5.91a}$$

$$y_1 = \bar{C}_c \bar{x}_c \tag{5.91b}$$

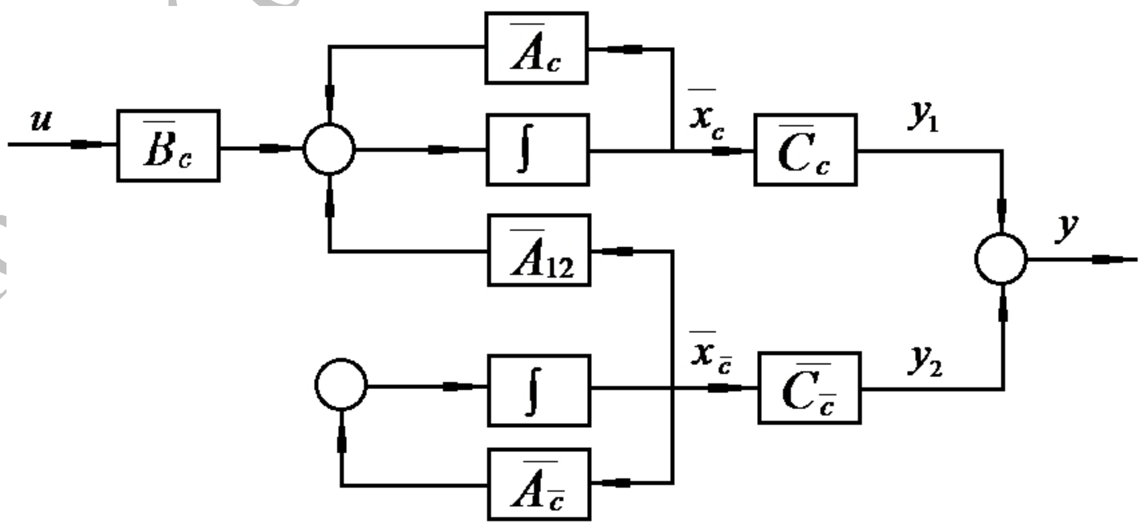

Figure5.3 System decomposition in terms of controllability

State structure diagram of equation(5.90) is shown in figure5.3. Observing the diagram in figure5.3, we can know that control signal u does not affect non-controllable state $\bar{x}_{\bar{c}}$ of system.

The method of ensuring non-singular linear transform matrix P_c is as follows:

The rank of controllable judgment matrix S_c is $n_1 < n$, namely,

$$rankS_c = rank\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n_1 < n$$

We can ensure n_1 linearly independent column vectors p_1, p_2, $\cdots$ p_{n_1} and then we can arbitrarily select $(n-n_1)$ linearly independent column vectors p_{n_1+1}, p_{n_1+2}, $\cdots$, p_n which is not related to vectors p_1, p_2, $\cdots$ p_{n_1}. Hence, non-singular linear transform matrix P of controllability structure decomposition is as follows:

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_{n_1} & p_{n_1+1} & p_{n_1+2} & \cdots & p_n \end{bmatrix} \tag{5.92}$$

Example5.23 Please decompose the following dynamic equations according to controllability.

$$\dot{x} = \begin{bmatrix} -2 & 0 & 1 & -1 \\ -4 & -2 & 4 & -4 \\ -4 & 0 & 3 & -3 \\ 4 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 & -1 & 2 \end{bmatrix} x$$

Solution:

Controllable judgment matrix S_c is the following:

$$rankS_c = rank\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = rank\begin{bmatrix} 2 & -2 & 2 & -2 \\ 1 & -2 & 4 & -8 \\ 2 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 = n_1 < n = 4$$

Obviously, the system is not completely controllable. We can select two linear-independence column vectors from controllable judgment matrix S_c:

$$p_1 = \begin{bmatrix} 2 & 1 & 2 & 0 \end{bmatrix}^T, \qquad p_2 = \begin{bmatrix} -2 & -2 & -2 & 0 \end{bmatrix}^T$$

And then we need to select two other linear independence column vectors to form non-singular transform matrix P_c of controllable structure decomposition.

$$P_c = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse matrix is

$$P_c^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1/2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, system matrix of controllability is the following:

$$\overline{A} = P_c^{-1} A P_c = \begin{bmatrix} 0 & -2 & -3 & 3 \\ 1 & -3 & -3.5 & 3.5 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B} = P_c^{-1} B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{C} = C P_c = \begin{bmatrix} 1 & -2 & -1 & 2 \end{bmatrix}$$

Apparently, controllable subsystem is the following:

$$\overline{A_c} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \qquad \overline{B_c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \overline{C_c} = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

We easily testify that above subsystem is controllable.

In order to find n_1 linear-independence column vectors, we need to do elementary transformation for controllable judgment matrix S_c:

$$S_c = \begin{bmatrix} 2 & -2 & 2 & -2 \\ 1 & -2 & 4 & -8 \\ 2 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & -4 & -7 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = S_c^{'}$$

Obviously, we can easily find that two former column vectors is linearly independent. At this time we need to select two other linear-independence column vectors to construct a non-singular linear transform matrix P_c of controllable decomposition:

$$P_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P_c^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{A} = P_c^{-1} A P_c = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -2 & 4 & -4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B} = P_c^{-1} B = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{C} = C P_c = \begin{bmatrix} 0 & 1 & -1 & 2 \end{bmatrix}$$

Hence controllable subsystem is the following:

$$\overline{A_c} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \overline{B_c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \overline{C_c} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Summarizing above example5.23, we can know that selecting linear-independence column vector is not unique and that corresponding canonical decomposition is also unique. However, these canonical decomposing systems could be transformed into each other for they have same eigenvalues; hence these canonical decomposing systems have same transfer function.

5.7.2 Canonical decomposition of observability

If system in equation(5.89) is not completely observable,namely $rankS_c = n_1 < n$, non-singular linear transform $x = P_o \bar{x}$ makes system in equation(5.89) become the following form:

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u , \qquad y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} \tag{5.92}$$

In equation(5.92), n_2 -dimension subsystem is observable, which could be expressed as follows:

$$\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u \tag{5.93a}$$

$$y_1 = \bar{C}_o \bar{x}_o \tag{5.93b}$$

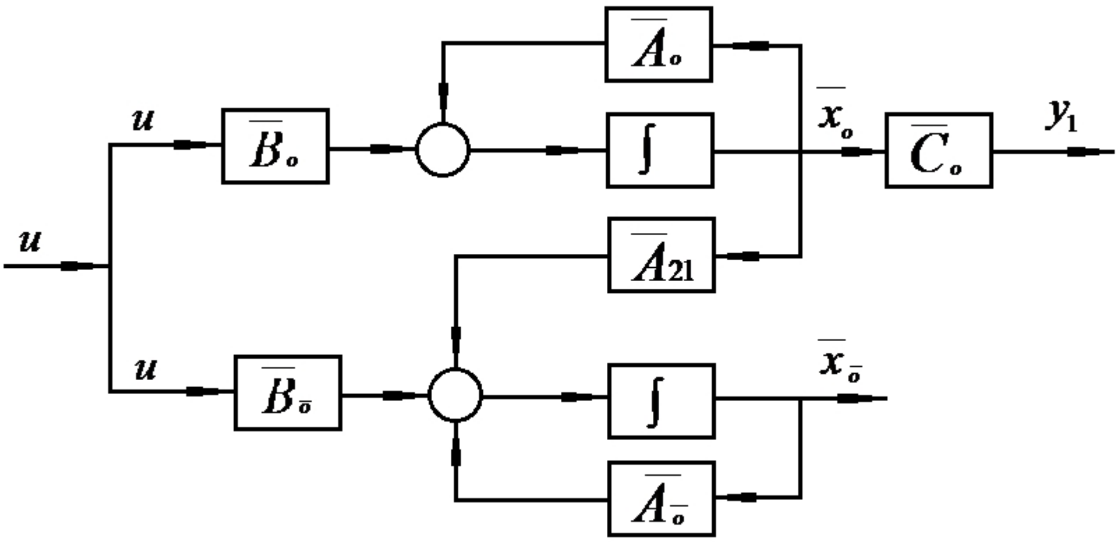

Figure5.4 System decomposition in terms of observability

State structure diagram in equation(5.92) is shown in figure5.4. Observing this diagram in figure5.4, we can easily find such information: the eigenvalues of new system matrix $\bar{A}_{\bar{o}}$ are eigenvalues which are not observable, but its movement information is reflected in output information.

The method of ensuring non-singular transform matrix P_o is the following:

As we know, the rank of observable judgment matrix S_o is less than maximum order, namely ,

$$rankS_o = rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n_2 < n$$

Hence, we can choose n_2 linear-independence row vectors $p_1 \quad p_2 \quad \cdots \quad p_{n2}$ in observable judgment matrix S_o, and then arbitrarily select $(n-n_2)$ linear-independence row vectors $p_{n2+1} \quad p_{n2+2} \quad \cdots \quad p_n$ to form non-linear transform matrix P_o:

$$P_o = \begin{bmatrix} p_1 \\ \vdots \\ p_{n2} \\ \vdots \\ p_n \end{bmatrix}^{-1} \tag{5.94}$$

Example5.24 Please decompose the system in example5.23 in terms of observability.

Solution:

Ensure the rank of observable judgment matrix S_o to do observable structure decomposition.

$$rankS_o = rank\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = rank\begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -2 & 2 & 0 \\ 4 & 4 & -4 & 4 \\ -8 & -8 & 8 & -4 \end{bmatrix} = rank\begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= rank\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 = n_2 < n = 4$$

here, we select two linear-independence unit row vectors to form non-singular transform matrix P_o:

$$P_o = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence, new system in terms of observable decomposition is the following:

$$\bar{A} = P_o^{-1}AP_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ -1 & -1 & -1 & 1 \\ -4 & -4 & 0 & 2 \end{bmatrix}, \quad \bar{B} = P_o^{-1}B = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{C} = CP_o = \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}$$

We can easily find relatively observable subsystem:

$$\bar{A}_o = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix}, \quad \bar{B}_o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C}_o = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

5.7.3 Decomposition of controllability and observability

If system in equation(5.89) is not completely controllable and not completely observable, namely $rankS_c = n_1 < n$ and $rankS_o = n_2 < n$, a non-singular linear transform $x = P\bar{x}$ makes system become the following form:

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u \tag{5.95a}$$

$$y = \begin{bmatrix} \bar{C}_{co} & \bar{C}_{c\bar{o}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} \tag{5.95b}$$

In equation(5.95), the significance of relative state vectors are as follows:

$\bar{x}_{co}$ represents n_1 -dimension controllable and observable state vector; $\bar{x}_{c\bar{o}}$ represents n_2 -dimension controllable but unobserved state vector; $\bar{x}_{\bar{c}o}$ represents n_3 -dimension uncontrollable but observable state vector; $\bar{x}_{\bar{c}\bar{o}}$ represents n_4 -dimension uncontrollable and unobserved state vector. Furthermore, such expression is always found: $n_1 + n_2 + n_3 + n_4 = n$. Hence, for no completely controllable and no completely observable system, it can be divided into four subsystems: controllable and observable subsystem, uncontrollable and observable subsystem, controllable and unobserved subsystem, uncontrollable and unobserved subsystem.

We can obviously find that the transfer function that we acquire is the transfer function of controllable and observable subsystem, namely:

$$G(s) = C(sE - A)^{-1}B = \bar{C}\left(sE - \bar{A}_{co}\right)^{-1}\bar{B}_{co} \tag{5.96}$$

Hence, system transfer function only depicts controllable and observable parts in this system, however other three parts are not depicted in system transfer function. Then we can draw such conclusion: state space could completely and thoroughly depicts system characteristics, but system transfer function can not completely include all information of system.

How do we ensure the non-singular transform matrix P in terms of controllable and observable decomposition? There are a lot of methods of realizing this aim. One of these methods is introduced here. Original system should be firstly decomposed into controllable subsystem and uncontrollable subsystem in terms of system controllability, then the two subsystems are decomposed again according to observability; finally all state vectors are rearranged to make system state equation become the form of equation(5.95).

The specific course is provided here.

Assume that the dynamic equation of linear time-invariant system is as follows:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.97}$$

Firstly, we decompose equation(5.97) in terms of system controllability.

In terms of controllable decomposition, we assume that the corresponding transform matrix P_c could be gained as follows:

$$\begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} = P_c \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix}, \text{ or } \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} = P_c^{-1} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} \tag{5.98}$$

Equation(5.98) is substituted into equation(5.97), and we can gain

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = P_c^{-1}AP_c\bar{x} + P_c^{-1}Bu = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + P_c^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \tag{5.99a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = CP_c\bar{x} = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} \tag{5.99b}$$

Hence, we can acquire the following two subsystems:

Controllable subsystem1:

$$\dot{\bar{x}}_c = P_c^{-1}AP_c\bar{x}_c + P_c^{-1}Bu = \bar{A}_c\bar{x}_c + \bar{A}_{12}\bar{x}_{\bar{c}} + P_c^{-1}\bar{B}_c u \tag{5.100a}$$

$$y_1 = CP_c\bar{x}_c = \bar{C}_c\bar{x}_c \tag{5.100b}$$

Uncontrollable subsystem2:

$$\dot{\bar{x}}_{\bar{c}} = \bar{A}_{\bar{c}}\bar{x}_{\bar{c}} \tag{5.101a}$$

$$y_2 = \bar{C}_{\bar{c}}\bar{x}_{\bar{c}} \tag{5.101b}$$

Then we canonically decompose above two subsystems in terms of system observability.

Similarly, we can assume the corresponding non-singular transform matrices P_{co} and $P_{\bar{c}o}$, and then get the following transform relations:

$$\bar{x}_c = P_{co}\tilde{x}_{co} \tag{5.102}$$

$$\bar{x}_{\bar{c}} = P_{\bar{c}o}\tilde{x}_{\bar{c}o} \tag{5.103}$$

By equation(5.103), uncontrollable subsystem could be decomposed in terms of system observability:

$$\begin{bmatrix}\dot{\tilde{x}}_{\bar{c}o}\\ \dot{\tilde{x}}_{\bar{c}\bar{o}}\end{bmatrix} = P_{\bar{c}o}^{-1}\bar{A}_{\bar{c}}P_{\bar{c}o}\begin{bmatrix}\tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} = \begin{bmatrix}\bar{A}_{\bar{c}o} & 0\\ \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}}\end{bmatrix}\begin{bmatrix}\tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} \tag{5.104a}$$

$$y_2 = \bar{C}_{\bar{c}}P_{\bar{c}o}\begin{bmatrix}\tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} = \begin{bmatrix}\bar{C}_{\bar{c}o} & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} \tag{5.104b}$$

By equation(5.102), controllable subsystem could be decomposed in terms of system observability:

$$\begin{bmatrix}\dot{\tilde{x}}_{co}\\ \dot{\tilde{x}}_{c\bar{o}}\end{bmatrix} = P_{co}^{-1}P_c^{-1}AP_cP_{co}\tilde{x}_{co} + P_{co}^{-1}P_c^{-1}Bu = P_{co}^{-1}\bar{A}_cP_{co}\tilde{x}_{co} + P_{co}^{-1}\bar{A}_{12}P_{\bar{c}o}\tilde{x}_{\bar{c}o} + P_{co}^{-1}P_c^{-1}\bar{B}_cu$$

$$= \begin{bmatrix}\bar{A}_{co} & 0\\ \bar{A}_{21} & \bar{A}_{c\bar{o}}\end{bmatrix}\begin{bmatrix}\tilde{x}_{co}\\ \tilde{x}_{c\bar{o}}\end{bmatrix} + \begin{bmatrix}\bar{A}_{13} & 0\\ \bar{A}_{23} & \bar{A}_{24}\end{bmatrix}\begin{bmatrix}\tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} + \begin{bmatrix}\bar{B}_{co}\\ \bar{B}_{c\bar{o}}\end{bmatrix}u \tag{5.105b}$$

$$y_1 = CP_c\bar{x}_c = \bar{C}_c\bar{x}_c = \bar{C}_cP_{co}\tilde{x}_{co} = \begin{bmatrix}\bar{C}_{co} & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_{co}\\ \tilde{x}_{c\bar{o}}\end{bmatrix} \tag{5.105b}$$

Summarizing above results of canonical decomposition, we can gain the following canonical decomposition of controllability and observability.

$$\begin{bmatrix}\dot{\tilde{x}}_{co}\\ \dot{\tilde{x}}_{c\bar{o}}\\ \dot{\tilde{x}}_{\bar{c}o}\\ \dot{\tilde{x}}_{\bar{c}\bar{o}}\end{bmatrix} = \begin{bmatrix}\bar{A}_{co} & 0 & \bar{A}_{13} & 0\\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24}\\ 0 & 0 & \bar{A}_{\bar{c}o} & 0\\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}}\end{bmatrix}\begin{bmatrix}\tilde{x}_{co}\\ \tilde{x}_{c\bar{o}}\\ \tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} + \begin{bmatrix}\bar{B}_{co}\\ \bar{B}_{c\bar{o}}\\ 0\\ 0\end{bmatrix}u \tag{5.106a}$$

$$y = \begin{bmatrix}\bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_{co}\\ \tilde{x}_{c\bar{o}}\\ \tilde{x}_{\bar{c}o}\\ \tilde{x}_{\bar{c}\bar{o}}\end{bmatrix} \tag{5.106b}$$

Example5.25 Please decompose the system in example5.23 in terms of controllability and observability.

Solution:

In example5.23, we have gotten structure decomposition of system in terms of controllability which are as follows:

$$\bar{A} = P_c^{-1} A P_c = \begin{bmatrix} 0 & -2 & -3 & 3 \\ 1 & -3 & -3.5 & 3.5 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{B} = P_c^{-1} B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C} = C P_c = \begin{bmatrix} 1 & -2 & -1 & 2 \end{bmatrix}$$

Through controllable decomposition, original system is divided into two controllable and uncontrollable subsystems:

Controllable subsystem:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ -3.5 & 3.5 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Uncontrollable subsystem:

$$\begin{bmatrix} \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}, \quad y_2 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$$

In the following, two new subsystems are decomposed again in terms of system observability.

For uncontrollable subsystem,

$$rankS_{\bar{c}o} = rank \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = 1 < 2$$

Hence, uncontrollable subsystem is not completely observable. We should select non-singular linear transform matrix $P_{\bar{c}o}$ to realize observable decomposition.

$$\begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = P_{\bar{c}o} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

Transform matrix $P_{\bar{c}o}$:

$$P_{\bar{c}o} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

Hence, observable structure decomposition of uncontrollable subsystem is the following:

$$\bar{A}_{\bar{c}o} = P_{\bar{c}o}^{-1} \bar{A}_{\bar{c}} P_{\bar{c}o} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{C}_{\bar{c}o} = C_{\bar{c}} P_{\bar{c}o} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Namely,

$$\begin{bmatrix} \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

For controllable subsystem, we should also judge the system observability.

$$rankS_{co} = rank \begin{bmatrix} \overline{C}_c \\ \overline{C}_c \overline{A}_c \end{bmatrix} = rank \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = 1 < 2$$

Hence, controllable subsystem is also completely observable. Similarly, a non-singular linear transform matrix P_{co} needs to be found to be decomposed according to observability.

We can select such transform matrix P_{co}:

$$\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = P_{co} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \qquad P_{co} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Then,

$$\overline{A}_{co} = P_{co}^{-1} \overline{A}_c P_{co} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\overline{A}_{13} = P_{co}^{-1} \overline{A}_{13} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ -3.5 & 3.5 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -3.5 & 3.5 \end{bmatrix},$$

$$\overline{C}_{co} = \overline{C}_c P_{co} = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Observable structure decomposition of controllable subsystem is the following:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ -3.5 & 3.5 \end{bmatrix} \begin{bmatrix} \overline{x}_3 \\ \overline{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

From the observable transform of uncontrollable subsystem, we can know

$$\begin{bmatrix} \overline{x}_3 \\ \overline{x}_4 \end{bmatrix} = P_{co} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

Above transform relationship is substituted into the observable structure decomposition of controllable subsystem, and then we can get:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ -3.5 & 3.5 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ -3.5 & 3.5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} -4 & 4 \\ 3.5 & -3.5 \end{bmatrix} \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

Summarizing above decomposition results, we can get that the canonical decomposition of system is the following:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 & 4 \\ 0 & -1 & 3.5 & -3.5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

Example5.26 Please decompose the following system in terms of controllability and observability.

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x$$

Solution:

Judge the controllability of system; and then we can acquire:

$$rankS_c = rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} = rank\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix} = 2 < n = 3$$

Hence, such system is not completely controllable.

Construct the non-singular transform matrix P_c:

$$P_c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then we can gain structure decomposition of system in terms of controllability which are as follows:

$$\overline{A}_c = P_c^{-1}AP_c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P_c^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad CP_c = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}$$

Then we can get two subsystems: controllable subsystem and uncontrollable subsystem.

Controllable subsystem:

$$\dot{\overline{x}}_c = \begin{bmatrix} \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \overline{x}_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}$$

Uncontrollable subsystem:

$$\dot{\overline{x}}_{\bar{c}} = \dot{\overline{x}}_3 = -\overline{x}_3, \quad y_2 = -2\overline{x}_3$$

In the following, we will need to judge system observability for uncontrollable system.

$$rankS_{\bar{c}o} = rank\begin{bmatrix} \overline{C}_{\bar{c}} \end{bmatrix} = rank\begin{bmatrix} -2 \end{bmatrix} = 1$$

Hence, uncontrollable subsystem is completely observable, and there is no structure decomposition for uncontrollable subsystem.

For controllable subsystem, we can gain its observable judgment matrix S_{co}:

$$rankS_{co} = rank\begin{bmatrix} \overline{C}_c \\ \overline{C}_c\overline{A}_c \end{bmatrix} = rank\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1 < n_c = 2$$

Hence, controllable subsystem is not observable, and we need to select non-singular transform matrix P_{co}:

$$P_{co} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then observable structure decomposition of controllable subsystem is as follows:

$$\overline{A}_{co} = P_{co}^{-1}\overline{A}_c P_{co} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$P_{co}^{-1}\overline{A}_{12}P_{co} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$, \quad P_{co}^{-1}\overline{B}_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overline{C}_{co} = \overline{C}_c P_{co} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Namely,

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tilde{x}_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u , \quad y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

Summarizing above results, we can gain the following canonical decomposition of system:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

5.8 Realization of Transfer Function Matrix

Transfer function which demonstrates the information transfer relationship between system input and system input only shows the dynamic action of controllable and observable subsystems of system. For transfer function matrix, there are infinite state space expressions; that's to say, a transfer function could depict infinite systems with different inner structure. In infinite systems with different inner structures, the system with minimum dimensions is the problem of minimum realization.

5.8.1 Basic conception of realization problem

If a state space could be depicted by the following expression

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{5.107}$$

The transfer function matrix of equation(5.107) could be gained as follows:

$$G(s) = C(sE - A)^{-1} B + D \tag{5.108}$$

State space expression in equation(5.107) is called the realization of transfer function matrix $G(s)$. We have deduced expression in equation(2.176). Here, we want to point out that no any a transfer function matrix finds corresponding realization. Transfer function matrix $G(s)$ needs to satisfy the objective condition of realization, namely:

(1) In each transfer function unit of transfer function matrix, polynomial coefficients

of denominator and numerator are both real constants.

(2) In each transfer function unit, the order number of denominator polynomial is less than that of numerator polynomial.

When the order number of denominator polynomial is equal to that of numerator polynomial, we can gain the following expression:

$$D = \lim_{s \to 0} G(s) \tag{5.109}$$

In terms of equation(5.108), we can gain the realization of state space $(A, \quad B, \quad C)$:

$$C(sE - A)^{-1} B = G(s) - D \tag{5.110}$$

5.8.2 Realization of controllable canonical form

We assume that transfer function matrix of SISO system is the following

$$G(s) = \frac{\bar{b}_n s^n + \bar{b}_{n-1} s^{n-1} + \cdots + \bar{b}_1 s + \bar{b}_0}{\bar{a}_n s^n + \bar{a}_{n-1} s^{n-1} + \cdots + \bar{a}_1 s + \bar{a}_0} \tag{5.111}$$

If there is no common factor between denominator polynomials and numerator polynomials, or if there is no appearance of counteraction between them, expression in equation(5.111) could be converted into the following form:

$$G(s) = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0} + d \tag{5.112}$$

For SISO system, we can gain the following expression from equation(5.112):

$$Y(s) = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0} U(s) + dU(s) \tag{5.113}$$

We order the following expressions could be found:

$$\begin{aligned} Y_1(s) &= \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0} U(s) \\ &= \frac{1}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0} \left(b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0\right) U(s) \end{aligned} \tag{5.114a}$$

$$Y_2(s) = dU(s) \tag{5.114b}$$

The corresponding block diagram is shown in figure5.5.

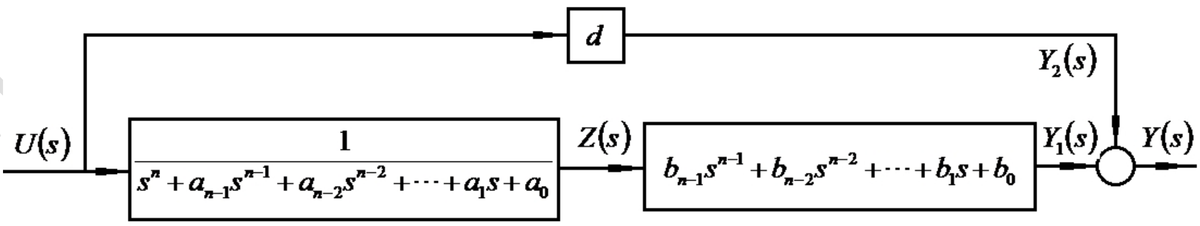

Figure5.5 Equivalent transformation of equation(5.114)

New variable $Z(s)$ is introduced to transform transfer function into differential equation group:

$$\begin{cases} z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_1 \dot{z} + a_0 = u \\ y_1 = b_{n-1}z^{(n-1)} + \cdots + b_1 \dot{z} + b_0 \\ y_2 = du \\ y = y_1 + y_2 \end{cases} \tag{5.115}$$

Take a series of state variables, $x_1 = z,\quad x_2 = \dot{z},\quad x_3 = \ddot{z},\quad \cdots,\quad x_n = z^{(n-1)}$, and then we could acquire the following state equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n = z^{(n)} &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + u \end{aligned} \tag{5.116}$$

Equation(5.116) could be expressed as matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{5.117a}$$

$$y = b_0x_1 + b_1x_2 + \cdots + b_{n-1}x_n + du = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x + du \tag{5.117b}$$

Equation(5.117) is called the realization of controllable canonical form.

Example5.27 Please gain the realization of controllable canonical form on the following system.

$$G(s) = \frac{s^2 + 4s + 6}{s^3 + +2s^2 + 6s + 10}$$

Solution:

In terms of equation(5.117), we can gain the realization of controllable canonical form is such:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -6 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 4 & 6 \end{bmatrix} x$$

Here, we should note that no all transfer functions are transformed into the realization of controllable canonical form. Only controllable system could be transformed into the realization of controllable canonical form. However its form is not only limited to above form.

5.8.3 Realization of observable canonical form

Transfer function in equation(5.112) is marked as follows:

$$G(s)=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1s+a_0}+d=G_1(s)+G_2(s) \tag{5.117}$$

Here,

$$G_1(s)=\frac{Y_1(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1s+a_0}, \quad G_2(s)=\frac{Y_2(s)}{U(s)}=d$$

Differential equation of transfer function $G_1(s)$ can be depicted as follows:

$$y_1^{(n)}+a_{n-1}y_1^{(n-1)}+\cdots+a_1\dot{y}_1+a_0y_1=b_{n-1}u^{(n-1)}+\cdots+b_1\dot{u}+b_0u \tag{5.118}$$

Initial condition is $y_1^{(n-1)}(0), \quad \cdots, \quad \dot{y}_1(0), \quad y_1(0); \; u^{(n-1)}(0), \quad \cdots, \quad \dot{u}(0), \quad u(0)$.

Laplace transform is exerted onto equation(5.118) and then we can gain

Selecting state variables is as follows:

$$\begin{aligned}
x_n &= y_1 \\
x_{n-1} &= \dot{y}_1+a_{n-1}y_1-b_{n-1}u=\dot{x}_n+a_{n-1}x_n-b_{n-1}u \\
x_{n-2} &= \ddot{y}_1+a_{n-1}\dot{y}_1-b_{n-1}\dot{u}-b_{n-2}u=\dot{x}_{n-1}+a_{n-2}x_n-b_{n-2}u \\
&\vdots \\
x_1 &= y_1^{(n-1)}+a_{n-1}y_1^{(n-2)}-b_{n-1}u^{(n-2)}+a_{n-2}y_1^{(n-3)}-b_{n-2}u^{(n-3)}+\cdots+a_1y_1-b_1u \\
&= \dot{x}_2+a_1x_n-b_1u
\end{aligned} \tag{5.119}$$

Obviously, under the condition of given transfer function and input signal $u(t)$, output signal $Y(s)$ could be ensured if initial state is given; then dynamic characteristic could be uniquely ensured. Hence above variables selected could satisfy the condition of state variables and they could be considered as state variables.

$$\begin{aligned}
\dot{x}_2 &= x_1-a_1x_n+b_1u \\
\dot{x}_3 &= x_2-a_2x_n+b_2u \\
&\vdots \\
\dot{x}_{n-1} &= x_{n-2}-a_{n-2}x_n+b_{n-2}u \\
\dot{x}_n &= x_{n-1}-a_{n-1}x_n+b_{n-1}u
\end{aligned}$$

Furthermore,

$$\dot{x}_1 = y_1^{(n)} + a_{n-1}y_1^{(n-1)} - b_{n-1}u^{(n-1)} + a_{n-2}y_1^{(n-2)} - b_{n-2}u^{(n-2)} + \cdots + a_1\dot{y}_1 - b_1\dot{u}$$

$$= \left(y_1^{(n)} + a_{n-1}y_1^{(n-1)} + a_{n-2}y_1^{(n-2)} + \cdots + a_1\dot{y}_1 + a_0 y_1\right)$$

$$-\left(b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + b_{n-3}u^{(n-3)} + \cdots + b_1\dot{u} + b_0 u\right) - a_0 y_1 + b_0 u$$

$$= -a_0 y_1 + b_0 u$$

$$= -a_0 x_n + b_0 u$$

Hence, we can gain the following state equation:

$$\begin{cases} \dot{x}_1 = -a_0 x_n + b_0 u \\ \dot{x}_2 = x_1 - a_1 x_n + b_1 u \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} - a_{n-1}x_n + b_{n-1}u \\ \dot{x}_n = x_{n-1} - a_n x_n + b_n u \end{cases} \tag{5.119a}$$

Or,

$$\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} x + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \tag{5.119b}$$

Output equation is as follows:

$$y = y_1 + y_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} x + du \tag{5.119c}$$

Above realization is called the realization of observable canonical form.

We could easily find that the coefficient matrices between controllable canonical form and observable canonical form transpose each other. Namely, the following expressions are always found:

$$A_c = A_o^T, \quad B_c = C_o^T, \quad C_c = B_o^T \tag{5.120}$$

Above characteristic is called **dual characteristic**.

Example5.28 Please acquire the realization of observable canonical form on the following system.

$$G(s) = \frac{s^2 + 4s + 6}{s^3 + +2s^2 + 6s + 10}$$

Solution:

In terms of the solution of example5.27, we can know these coefficient matrices:

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -6 & -2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 4 & 6 \end{bmatrix}$$

From the relationship shown in equation(5.20), we can acquire the following observable canonical form:

$$\dot{x}=\begin{bmatrix}0 & 0 & -10\\ 1 & 0 & -6\\ 0 & 1 & -2\end{bmatrix}x+\begin{bmatrix}1\\ 4\\ 6\end{bmatrix}u, \qquad y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

5.8.4 Realization of diagonal canonical form

We assume that system transfer function is the following:

$$G(s)=\frac{N(s)}{D(s)} \tag{5.121}$$

Denominator expression in above equation could be expressed as follows:

$$D(s)=(s-s_1)(s-s_2)(s-s_3)\cdots\cdots(s-s_n)$$

If poles of equation(5.121) is $s_i(i=1,2,\cdots n)$, equation(5.121) could be decomposed into the following form in terms of partial fraction theory.

$$G(s)=\frac{C_1}{s-s_1}+\frac{C_2}{s-s_2}+\cdots\cdots+\frac{C_n}{s-s_n} \tag{5.122}$$

Among equation(5.122), coefficient C_i can be calculated by the following expression:

$$C_i=\lim_{s\to s_i}(s-s_i)G(s)=\frac{N(s_i)}{\dot{D}(s_i)} \tag{5.123}$$

Hence, output signal $Y(s)$ could be expressed as the following form:

$$Y(s)=G(s)U(s)=\sum_{i=1}^{n}\frac{C_i}{s-s_i}U(s)=\sum_{i=1}^{n}C_i\frac{U(s)}{s-s_i}=\sum_{i=1}^{n}C_iX_i(s) \tag{5.124}$$

Here,

$$X_i(s)=\frac{U(s)}{s-s_i} \tag{5.125}$$

Then output signal $Y(s)$ could be transformed into the following form:

$$Y(s)=\sum_{i=1}^{n}C_iX_i(s) \tag{5.126}$$

We can gain such expression from equation(5.125):

$$sX_i(s)-s_iX_i(s)=U(s) \tag{5.127}$$

Corresponding differential equation is

$$\dot{x}_i=s_ix_i+u; \quad i=1,2,\cdots,n \tag{5.128}$$

Obviously, when initial state $x_i(0)$ and input signal $u(t)$ are given, the output signal $y(t)$ would be uniquely confirmed. Hence, we can regard the group variable x_i as a group of system state variable. System state equation from equation(5.122) could be expressed as the following form:

$$\begin{cases} \dot{x}_1 = s_1 x_1 + u \\ \dot{x}_2 = s_2 x_2 + u \\ \quad\vdots \\ \dot{x}_n = s_n x_n + u \end{cases} \tag{5.129a}$$

Or,

$$\dot{x} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u \tag{5.129b}$$

Output equation of system is

$$y = \sum_{i=1}^{n} C_i x_i(t) = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} x \tag{5.129c}$$

All expressions in equation(1.29) are called the realization of diagonal canonical form. We can easily find that system matrix A is a diagonal matrix which factors are roots of system characteristic equation. Input matrix B is a column vector which all factors are one; and output matrix C is a row vector which factors are the numerator coefficients in equation(5.122).

Example5.29 Please acquire the realization of diagonal canonical form on the following system.

$$G(s) = \frac{s^2 + 3s + 5}{s^3 + 7s^2 + 14s + 8}$$

Solution:

$$D(s) = s^3 + 7s^2 + 14s + 8 = (s+1)(s+2)(s+4)$$

Hence, system transfer function could be transformed into the following fractions:

$$G(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{s+4}$$

In terms of equation(5.123), the coefficients of fraction could be gained:

$$C_1 = \lim_{s \to -1} \frac{s^2 + 3s + 5}{(s+2)(s+4)} = 1, \quad C_2 = \lim_{s \to -2} \frac{s^2 + 3s + 5}{(s+1)(s+4)} = -\frac{3}{2}, \quad C_3 = \lim_{s \to -4} \frac{s^2 + 3s + 5}{(s+1)(s+2)} = \frac{3}{2}$$

The realization of diagonal canonical form is the following:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} x$$

5.8.5 Realization of Jordan canonical form

If system transfer function has several same poles, it would be realized by Jordan canonical form. In order to explore the case of r multiple roots and that other roots are single root, we assume that the fraction of transfer function is the following:

$$G(s)=\sum_{i=1}^{r}\frac{C_i}{(s-s_1)^i}+\sum_{i=r+1}^{n}\frac{C_i}{s-s_i} \tag{5.130}$$

Fraction coefficient C_i could be acquired by residue theorem.

$$C_i=\frac{1}{(r-i)!}\lim_{s\to s_i}\frac{d^{(r-i)}}{ds^{(r-i)}}\left[(s-s_i)^r G(s)\right] \tag{5.131}$$

$$Y(s)=\sum_{i=1}^{r}C_i\frac{U(s)}{(s-s_i)^i}+\sum_{i=r+1}^{n}C_i\frac{U(s)}{s-s_i}$$

Order

$$X_i(s)=\frac{U(s)}{(s-s_i)^{r-i+1}};\quad i=1,2,\cdots r \tag{5.132a}$$

$$X_i(s)=\frac{U(s)}{s-s_i};\quad i=r+1,r+2,\cdots n \tag{5.132b}$$

When $i=r$,

$$X_r(s)=\frac{U(s)}{s-s_1}$$

Its differential equation is $\dot{x}_r-s_1x_r=u$; hence when initial state $x_r(0)$ is given, system state $x_r(t)$ could be uniquely confirmed.

For,

$$X_i(s)=\frac{U(s)}{(s-s_i)^{r-i+1}};\quad i=1,2,\cdots r$$

then,

$$X_{i+1}(s)=\frac{U(s)}{(s-s_i)^{r-i}};\quad i=1,2,\cdots r-1$$

$$X_i(s)=\frac{1}{s-s_i}X_{i+1}(s);\quad i=1,2,\cdots r-1$$

Corresponding differential equation is the following:

$$\dot{x}_i-s_ix_i=x_{i+1};\quad i=1,2,\cdots r-1$$

Order $i=r-1$,

$$\dot{x}_{r-1}-s_1x_{r-1}=x_r$$

System state $x_r(t)$ has been gained; hence when $x_{r-1}(0)$ is given, state $x_{r-1}(t)$

could be uniquely confirmed. In terms of similar methods, we can acquire a series of state $x_{r-2}(t),\quad x_{r-3}(t),\quad \cdots,\quad x_1(t)$.

Similar to the deduction of diagonal canonical form, we also have the following expression:

$$\dot{x}_i - s_i x_i = u;\quad i = r+1, r+2, \cdots n$$

When $x_i(0)$ is given, state $x_i(t)$ will be uniquely confirmed.

Summarizing above discussion, we can know that state $x_i(t)$ could be regarded as system state variable. Then state equation of equation(5.130) is the following:

$$\begin{cases} \dot{x}_1 = s_1 x_1 + x_2 \\ \dot{x}_2 = s_1 x_2 + x_3 \\ \quad\vdots \\ \dot{x}_{r-1} = s_1 x_{r-1} + x_r \\ \dot{x}_r = s_1 x_r + u \\ \dot{x}_{r+1} = s_{r+1} x_{r+1} + u \\ \quad\vdots \\ \dot{x}_n = s_n x_n + u \end{cases} \tag{5.133a}$$

Or,

$$\dot{x} = \begin{bmatrix} s_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & s_1 & 1 & 0 & & & & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & s_1 & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & s_1 & 0 & \ddots & \vdots \\ \vdots & & 0 & & 0 & s_{r+1} & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & s_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \tag{5.133b}$$

Output equation is such

$$Y(s) = \sum_{i=1}^{r} C_{r-i+1} X_i(s) + \sum_{i=r+1}^{n} C_i X_i(s) \tag{5.133c}$$

$$y = \sum_{i=1}^{r} C_{r-i+1} x_i + \sum_{i=r+1}^{n} C_i x_i = \begin{bmatrix} C_r & C_{r-1} & \cdots & C_1 & C_{r+1} & \cdots & C_n \end{bmatrix} x \tag{5.133d}$$

Example5.30 Please acquire the realization of Jordan canonical form on the following system.

$$G(s) = \frac{2s+1}{s^4 + 8s^3 + 21s^2 + 22s + 8}$$

Solution:

Factorize the denominator of system transfer function $G(s)$.

$$D(s) = s^4 + 8s^3 + 21s^2 + 22s + 8 = (s+1)^2(s+2)(s+4)$$

Obviously we can easily find that there are two multiple roots in $D(s)$ polynomial. And then partial fraction of system transfer function $G(s)$ is the following:

$$G(s) = \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{C_3}{s+2} + \frac{C_4}{s+4}$$

Apply residue theorem o above expression of system transfer function $G(s)$. And then we can acquire above undetermined coefficients C_i .

$$C_1 = \lim_{s\to -1} \frac{1}{(2-1)!}\frac{d}{ds}\left[(s+1)^2 G(s)\right] = \lim_{s\to -1} \frac{2(s^2+6s+8)-(2s+1)(2s+6)}{(s^2+6s+8)^2} = \frac{10}{3}$$

$$C_2 = \lim_{s\to -1} \frac{1}{(2-2)!}\left[(s+1)^2 G(s)\right] = \lim_{s\to -1} \frac{2s+1}{(s+2)(s+4)} = -\frac{1}{3}$$

$$C_3 = \lim_{s\to -2} (s+2)G(s) = \lim_{s\to -2} \frac{2s+1}{(s+1)^2(s+4)} = -\frac{3}{2}$$

$$C_4 = \lim_{s\to -4} (s+4)G(s) = \lim_{s\to -4} \frac{2s+1}{(s+1)^2(s+2)} = \frac{7}{18}$$

Hence, state equation of system transfer function is such:

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

Output equation is the following:

$$y = \begin{bmatrix} -\frac{1}{3} & \frac{10}{3} & -\frac{3}{2} & \frac{7}{18} \end{bmatrix} x$$

5.8.6 Minimum Realization

Transfer function only demonstrates the dynamic action of controllable and observable subsystem from whole system. For a realizable transfer function matrix, there will be infinite corresponding state space expressions. In the viewpoint of engineering, how to find such realization with minimum dimension will be very significant.

1.Minimum realization definition for state space expression

We assume that a realization of system transfer function $G(s)$ is the following:

$$\dot{x} = Ax + Bu \tag{5.134a}$$

$$y = Cx \tag{5.134b}$$

If there is no other realization of the following

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{5.135a}$$

$$y = \overline{C}\,\overline{x} \tag{5.135b}$$

Its dimension of state variable $\overline{x}$ is less than that of state variable x. Then the realization of equation(5.134) is called minimum realization of system transfer function $G(s)$.

System transfer function $G(s)$ only demonstrates the dynamic action of controllable and observable subsystem from whole system, hence some uncontrollable or unobserved state subsystems are eliminated and these eliminated subsystems don't affect system transfer function. That's to say, such system transfer function with uncontrollable or unobserved subsystem will make that some control system does not become a minimum realization. And then how do we perhaps find minimum realization of system?

2.How to find minimum realization from state space expression

A realization of system transfer function $G(s)$ is the following:

$$\dot{x} = Ax + Bu \tag{5.136a}$$

$$y = Cx + Du \tag{5.136b}$$

Then the sufficient and necessary condition of minimum realization is that *equation(5.136) is controllable and observable*. In terms of such recognition, we can ensure the minimum realization of any a system. The transfer function $G(s)$ of equation(5.136) as we have known in chapter2 is the following:

$$G(s) = C(sE - A)^{-1}B + D$$

If $D = 0$, we can get $G(s) = C(sE - A)^{-1}B$.

The common procedure of finding minimum realization is the following:

(1)For a given transfer function $G(s)$, firstly select a realization $\sum(A,\ \ B,\ \ C)$ of system; the simplest method is to select controllable canonical form or observable canonical form.

(2)For above original realization $\sum(A,\ \ B,\ \ C)$, decompose original system $\sum(A,\ \ B,\ \ C)$ in terms of controllable canonical form.

(3)Then on the basis of the decomposition of controllable canonical form, decompose system in terms of observable canonical form.

(4)Ensure controllable and observable subsystem, such subsystem is a minimum realization.

Example5.31 Please find the minimum realization of the following system.

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -4 & -2 & 1 & 2 \end{bmatrix} x$$

Solution:

According to given dynamic equations, we can gain:

System matrix A : $A=\begin{bmatrix}1 & 0 & 0 & -1\\0 & -3 & 0 & 0\\0 & 2 & 1 & 0\\0 & 0 & 0 & -2\end{bmatrix}$, input matrix B : $B=\begin{bmatrix}2\\1\\2\\0\end{bmatrix}$

Output matrix C : $C=\begin{bmatrix}-4 & -2 & 1 & 2\end{bmatrix}$

Step1, judge whether given system is controllable.

$$rankS_c = rank\begin{bmatrix}B & AB & A^2B & A^3B\end{bmatrix} = rank\begin{bmatrix}2 & 2 & 2 & 2\\1 & -3 & 9 & -27\\2 & 4 & -2 & 16\\0 & 0 & 0 & 0\end{bmatrix} = 2 = n_1 < 4 = n$$

Hence, given system is not completely controllable.

Step2, in terms of controllable decomposition, we need to acquire new system.

We may select two linearly independent column vector such as:

$$p_1 = \begin{bmatrix}2 & 1 & 2 & 0\end{bmatrix}^T, \qquad p_2 = \begin{bmatrix}2 & -3 & 4 & 0\end{bmatrix}^T$$

And then we select two other linearly independent column vector to construct non-singular transform matrix P_c :

$$P_c = \begin{bmatrix}2 & 2 & 0 & 0\\1 & -3 & 0 & 0\\2 & 4 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Inverse matrix of transform matrix P is the following

$$P_c^{-1} = \begin{bmatrix}0.375 & 0.25 & 0 & 0\\0.125 & -0.25 & 0 & 0\\-1.25 & 0.5 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Then in terms of controllable decomposition, we can get new dynamic equation:

$$\overline{A}_c = P_c^{-1}AP_c = \begin{bmatrix}0 & 3 & 0 & -0.375\\1 & -2 & 0 & -0.125\\0 & 0 & 1 & 1.25\\0 & 0 & 0 & -2\end{bmatrix}, \qquad \overline{B}_c = P_c^{-1}B = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\overline{C}_c = CP_c = \begin{bmatrix}-8 & 2 & 1 & 2\end{bmatrix}$$

Namely, new system equations are as follows:

$$\dot{\overline{x}} = \begin{bmatrix}0 & 3 & 0 & -0.375\\1 & -2 & 0 & -0.125\\0 & 0 & 1 & 1.25\\0 & 0 & 0 & -2\end{bmatrix}\overline{x} + \begin{bmatrix}1\\0\\0\\0\end{bmatrix}u, \quad y = \begin{bmatrix}-8 & 2 & 1 & 2\end{bmatrix}\overline{x}$$

Step3, we continue to decompose the new system in terms of observable characteristic of system. Controllable subsystem is the following:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -0.375 \\ 0 & -0.125 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} -8 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Uncontrollable subsystem is the following:

$$\begin{bmatrix} \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1.25 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}, \qquad y_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$$

Here we decompose the uncontrollable subsystem in terms of observable characteristic of system if this subsystem is not observable.

$$rank S_{\bar{c}o} = rank \begin{bmatrix} \bar{C}_{\bar{c}} \\ \bar{C}_{\bar{c}} \bar{A}_{\bar{c}} \end{bmatrix} = rank \begin{bmatrix} 1 & 2 \\ 1 & -2.75 \end{bmatrix} = 2$$

Hence uncontrollable subsystem is observable.

The judgment matrix S_{co} of controllable subsystem is as follows:

$$rank S_{co} = rank \begin{bmatrix} \bar{C}_c \\ \bar{C}_c \bar{A}_c \end{bmatrix} = rank \begin{bmatrix} -8 & 2 \\ 2 & -28 \end{bmatrix} = 2$$

Hence controllable subsystem is also observable.

Controllable and observable subsystem matrix $\bar{A}_{co}$:

$$\bar{A}_{co} = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$$

Controllable and observable input matrix $\bar{B}_{co}$:

$$\bar{B}_{co} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$

Controllable and observable output matrix $\bar{C}_{co}$:

$$\bar{C}_{co} = \begin{bmatrix} -8 & 2 \end{bmatrix}$$

Step4, testify above given matrices to ensure minimum realization of system.

$$rank \bar{S}_c = rank \begin{bmatrix} \bar{B}_{co} & \bar{A}_{co} \bar{B}_{co} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 = n_{co}$$

$$rank \bar{S}_o = rank \begin{bmatrix} \bar{C}_{co} \\ \bar{C}_{co} \bar{A}_{co} \end{bmatrix} = rank \begin{bmatrix} -8 & 2 \\ 2 & -28 \end{bmatrix} = 2 = n_{co}$$

Hence above controllable and observable subsystem $\sum \left(\bar{A}_{co}, \quad \bar{B}_{co}, \quad \bar{C}_{co} \right)$ is minimum realization which we find according to given transfer function in this example.

As we have known, system transfer function is that of controllable and observable system in new system. Hence system transfer function is as follows:

$$G(s)=C(sE-A)^{-1}B=\overline{C}_{co}\left(sE-\overline{A}_{co}\right)^{-1}\overline{B}_{co}$$

Here, we utilize recurrence method to gain the inverse matrix of expression $\left(sE-\overline{A}_{co}\right)^{-1}$.

$$B_1=E\,,\qquad a_1=-tr\left(\overline{A}_{co}B_1\right)=-tr\overline{A}_{co}=2$$

$$B_2=\overline{A}_{co}B_1+a_1E=\overline{A}_{co}+2E=\begin{bmatrix}2&3\\1&0\end{bmatrix},\quad a_2=-\frac{1}{2}tr\left(\overline{A}_{co}B_2\right)=-\frac{1}{2}tr\begin{bmatrix}3&0\\0&3\end{bmatrix}=-3$$

$$\left(sE-\overline{A}_{co}\right)^{-1}=\frac{B_1s+B_2}{s^2+a_1s+a_2}=\frac{1}{s^2+2s-3}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}s+\begin{bmatrix}2&3\\1&0\end{bmatrix}\right\}=\frac{1}{s^2+2s-3}\begin{bmatrix}s+2&3\\1&s\end{bmatrix}$$

$$G(s)=\overline{C}_{co}\left(sE-\overline{A}_{co}\right)^{-1}\overline{B}_{co}=\frac{1}{s^2+2s-3}\begin{bmatrix}-8&2\end{bmatrix}\begin{bmatrix}s+2&3\\1&s\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=-\frac{2(4s+7)}{s^2+2s-3}$$

3.How to find minimum realization from system transfer function matrix.

For multi-input and multi-ouput system, if it could be written into the form of single-input and single-output system, we can assume that the following expression of multi-input and multi-output system is as follows:

$$G(s)=\frac{\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+\cdots+\beta_1s+\beta_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}\tag{5.137}$$

In equation(5.137), β_{n-1}, β_{n-2}, $\cdots$, β_1, β_0 — $m\times r$ constant coefficient matrix, denominator polynomial is system characteristic polynomial.Obviously, transfer function in equation(5.137) is matrix of rational form. When $m=r$, transfer function in equation(5.137) will become a single-input and single-output transfer function.

For transfer function in equation(5.137), its controllable canonical form is as follows:

$$A_c=\begin{bmatrix}0_r&E_r&0_r&\cdots&0_r\\0_r&0_r&E_r&\cdots&0_r\\\vdots&\vdots&\vdots&\ddots&\vdots\\0_r&0_r&0_r&\cdots&E_r\\-a_0E_r&-a_1E_r&-a_2E_r&\cdots&-a_{n-1}E_r\end{bmatrix}\tag{5.138a}$$

$$B_c=\begin{bmatrix}0_r\\0_r\\\vdots\\0_r\\E_r\end{bmatrix}\tag{5.138b}$$

$$C_c=\begin{bmatrix}\beta_0&\beta_1&\cdots&\beta_{n-2}&\beta_{n-1}\end{bmatrix}\tag{5.138c}$$

In above controllable canonical form, 0_r and E_r are $r\times r$ zero matrix and unit matrix respectively; sign r is input dimension of transfer function $G(s)$; sign n is highest order of characteristic polynomial.

Similarly, the observable canonical form for equation(5.137) is as follows:

$$A_o=\begin{bmatrix} 0_m & 0_m & \cdots & 0_m & -a_0E_m \\ E_m & 0_m & \cdots & 0_m & -a_1E_m \\ 0_m & E_m & \cdots & 0_m & -a_2E_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_m & 0_m & \cdots & E_m & -a_{n-1}E_m \end{bmatrix} \tag{5.139a}$$

$$B_o=\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \tag{5.139b}$$

$$C_o=\begin{bmatrix} 0_m & 0_m & \cdots & E_m \end{bmatrix} \tag{5.139c}$$

Here, 0_m and E_m are $m\times m$ zero matrix and unit matrix respectively; sign m is output dimension.

After we gain the controllable or observable canonical form, we need to find the controllable and observable subsystem in relative canonical form. Finally ensure the minimum realization.

Note that the dimension of minimum realization has to be less than that of given system.

Example5.32 Please find the minimum realization of the following system.

$$G(s)=\begin{bmatrix} \frac{s+4}{s+1} & \frac{1}{s+3} \\ \frac{s}{s+1} & \frac{s+1}{s+2} \end{bmatrix}$$

Solution:

Step1 achieve the controllable canonical form of given transfer function matrix.

Convert the given transfer function matrix into a real fraction, namely

$$G(s)=\begin{bmatrix} \frac{s+4}{s+1} & \frac{1}{s+3} \\ \frac{s}{s+1} & \frac{s+1}{s+2} \end{bmatrix}=\begin{bmatrix} \frac{3}{s+1} & \frac{1}{s+3} \\ -\frac{1}{s+1} & -\frac{1}{s+2} \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

That's to say,

$$G(s)=C(sE-A)^{-1}B+D=\begin{bmatrix} \frac{3}{s+1} & \frac{1}{s+3} \\ -\frac{1}{s+1} & -\frac{1}{s+2} \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C(sE-A)^{-1}B=\begin{bmatrix}\frac{3}{s+1} & \frac{1}{s+3}\\ -\frac{1}{s+1} & -\frac{1}{s+2}\end{bmatrix},\qquad D=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Adjust and merge the polynomial of expression $C(sE-A)^{-1}B$ in the form of decreasing power.

$$C(sE-A)^{-1}B=\begin{bmatrix}\frac{3}{s+1} & \frac{1}{s+3}\\ -\frac{1}{s+1} & -\frac{1}{s+2}\end{bmatrix}=\frac{1}{s^3+6s^2+11s+6}\begin{bmatrix}3s^2+15s+18 & s^2+3s+2\\ -s^2-5s-6 & -s^2-4s-3\end{bmatrix}$$

$$=\frac{1}{s^3+6s^2+11s+6}\left\{\begin{bmatrix}3 & 1\\ -1 & -1\end{bmatrix}s^2+\begin{bmatrix}15 & 3\\ -5 & -4\end{bmatrix}s+\begin{bmatrix}18 & 2\\ -6 & -3\end{bmatrix}\right\}$$

Comparing with equation(5.137), we can acquire the following parameter matrices:

$$a_0=6,\quad a_1=11,\quad a_2=6$$

$$\beta_0=\begin{bmatrix}18 & 2\\ -6 & -3\end{bmatrix},\quad \beta_1=\begin{bmatrix}15 & 3\\ -5 & -4\end{bmatrix},\quad \beta_2=\begin{bmatrix}3 & 1\\ -1 & -1\end{bmatrix},\quad m=r=2$$

Above coefficients and coefficient matrices are substituted into equation(5.138) and then we can achieve controllable canonical form.

$$A_c=\begin{bmatrix}0_{2\times2} & E_{2\times2} & 0_{2\times2}\\ 0_{2\times2} & 0_{2\times2} & E_{2\times2}\\ -a_0E_{2\times2} & -a_1E_{2\times2} & -a_2E_{2\times2}\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -6 & 0 & -11 & 0 & -6 & 0\\ 0 & -6 & 0 & -11 & 0 & -6\end{bmatrix}\quad D=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$B_c=\begin{bmatrix}0_{2\times2}\\ 0_{2\times2}\\ E_{2\times2}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix},\quad C_c=[\beta_0\quad \beta_1\quad \beta_2]=\begin{bmatrix}18 & 2 & 15 & 3 & 3 & 1\\ -6 & -3 & -5 & -4 & -1 & -1\end{bmatrix}$$

Step2 judge whether above controllable canonical form is observable.

$$S_{co}=\begin{bmatrix}C_c\\ C_cA_c\\ C_cA_c^2\end{bmatrix}=\begin{bmatrix}18 & 2 & 15 & 3 & 3 & 1\\ -6 & -3 & -5 & -4 & -1 & -1\\ -18 & -6 & -15 & -9 & -3 & -3\\ 6 & 6 & 5 & 8 & 1 & 2\\ 18 & 18 & 15 & 27 & 3 & 9\\ -6 & -12 & -5 & -16 & -1 & -4\end{bmatrix}$$

For $rankS_{co}=3<6$, this controllable canonical form is not completely observable, namely, the controllable canonical form is not a minimum realization.

Step3 observably decompose the controllable canonical form $\sum(A_c,\ B_c,\ C_c)$.

Observable transform matrix P_o is selected as follows:

$$P_o=\begin{bmatrix}18&2&15&3&3&1\\-6&-3&-5&-4&-1&-1\\-18&-6&-15&-9&-3&-3\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}^{-1}=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&-1&0.3333&0&-1&0\\0.5&0&0.1667&-6&0&-5\\-0.5&3&-1.5&0&1&0\end{bmatrix}$$

Hence, observable canonical form is as follows:

$$\overline{A}=P_o^{-1}A_cP_o=\begin{bmatrix}0&0&1&0&0&0\\-0.5&-2&-0.1667&0&0&0\\-3&0&-4&0&0&0\\0&0&0&0&0&1\\0&-1&0.3333&0&-1&0\\0.5&0&0.1667&-6&0&-5\end{bmatrix}=\begin{bmatrix}\overline{A}_{co}&0\\\overline{A}_{21}&\overline{A}_{c\bar{o}}\end{bmatrix}$$

$$\overline{B}=P_o^{-1}B_c=\begin{bmatrix}3&1\\-1&-1\\-3&-3\\0&0\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}\overline{B}_{co}\\0\end{bmatrix},\quad \overline{C}=C_cP_o=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\end{bmatrix}=\begin{bmatrix}\overline{C}_{co}&0\end{bmatrix}$$

Then we can gain the controllable and observable subsystem $\sum(\overline{A}_{co},\ \overline{B}_{co},\ \overline{C}_{co})$:

$$\overline{A}_{co}=\begin{bmatrix}0&0&1\\-0.5&-2&-0.1667\\-3&0&-4\end{bmatrix},\quad \overline{B}_{co}=\begin{bmatrix}3&1\\-1&-1\\-3&-3\end{bmatrix},\quad \overline{C}_{co}=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

Step4 confirm the minimum realization of system.

$$rank\,\overline{S}_c=\begin{bmatrix}\overline{B}_{co}&\overline{A}_{co}\overline{B}_{co}&(\overline{A}_{co})^2\overline{B}_{co}\end{bmatrix}=rank\begin{bmatrix}3&1&-3&-3&3&9\\-1&-1&1.0001&2.0001&-1.0003&-4.0005\\-3&-3&3&9&-3&-27\end{bmatrix}=3$$

$$rank\,\overline{S}_o=rank\begin{bmatrix}\overline{C}_{co}\\\overline{C}_{co}\overline{A}_{co}\\\overline{C}_{co}(\overline{A}_{co})^2\end{bmatrix}=rank\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\-0.5&-2&-0.1667\\-3&0&-4\\1.5001&4&0.5002\end{bmatrix}=3$$

Hence, minimum realization of given system is subsystem $\sum\left(\overline{A}_{co},\quad \overline{B}_{co},\quad \overline{C}_{co}\right)$.

Minimum realization of system is such:

$$\overline{A}_{co}=\begin{bmatrix}0 & 0 & 1\\ -0.5 & -2 & -0.1667\\ -3 & 0 & -4\end{bmatrix},\ \overline{B}_{co}=\begin{bmatrix}3 & 1\\ -1 & -1\\ -3 & -3\end{bmatrix},\quad \overline{C}_{co}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ D=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Example5.33 Please find the observable canonical form of example5.32.

Solution:

In terms of results in example5.32, we can gain the following parameter matrices.

$$a_0=6,\quad a_1=11,\quad a_2=6$$

$$\beta_0=\begin{bmatrix}18 & 2\\ -6 & -3\end{bmatrix},\quad \beta_1=\begin{bmatrix}15 & 3\\ -5 & -4\end{bmatrix},\qquad \beta_2=\begin{bmatrix}3 & 1\\ -1 & -1\end{bmatrix},\ m=r=2$$

Comparing with equation(5.139), we can acquire the observable canonical form:

$$A_o=\begin{bmatrix}0_{2\times2} & 0_{2\times2} & -a_0E_{2\times2}\\ E_{2\times2} & 0_{2\times2} & -a_1E_{2\times2}\\ 0_{2\times2} & E_{2\times2} & -a_2E_{2\times2}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & -6 & 0\\ 0 & 0 & 0 & 0 & 0 & -6\\ 1 & 0 & 0 & 0 & -11 & 0\\ 0 & 1 & 0 & 0 & 0 & -11\\ 0 & 0 & 1 & 0 & -6 & 0\\ 0 & 0 & 0 & 1 & 0 & -6\end{bmatrix}_{6\times6}$$

$$B_o=\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}=\begin{bmatrix}18 & 2\\ -6 & -3\\ 15 & 3\\ -5 & -4\\ 3 & 1\\ -1 & -1\end{bmatrix}\qquad C_o=\begin{bmatrix}0_m & 0_m & E_m\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Of course, we can also utilize dual characteristic in equation(5.120) to gain above observable canonical form.

5.9 Pole Zero Cancellation and System Properties

As we have known, controllable and observable system and minimum realization have same significance. At this time, we perhaps ask whether we could judge the controllability and observability of given system by system transfer function. For SISO system, the sufficient and necessary condition of system controllability and observability is that there is no pole zero cancellation between numerator and denominator of system transfer function. For MIMO system, pole zero cancellation appears in system transfer function, but it is possible that system transfer function is controllable and observable. In this section, we will discuss the theme of pole zero cancellation and system properties.

For a SISO system $\sum(A, \quad B, \quad C)$, state space expression is the following:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.140}$$

The sufficient and necessary condition that system in equation(5.140) is controllable and observable is that there is no pole zero cancellation between numerator and denominator in transfer function $G(s)$.

$$G(s) = C(sE - A)^{-1}B \tag{5.141}$$

Demonstration:

Sufficient demonstration.

If system $\sum(A, \quad B, \quad C)$ is not a minimum realization of transfer function $G(s)$, there will be another system $\sum(\overline{A}, \quad \overline{B}, \quad \overline{C})$ which has less dimension.

$$\begin{cases} \dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u \\ y = \overline{C}\overline{x} \end{cases} \tag{5.142}$$

Furthermore the following expression is always found:

$$G(s) = C(sE - A)^{-1}B = \overline{C}(sE - \overline{A})^{-1}\overline{B} \tag{5.143}$$

For the order of new system matrix $\overline{A}$ is less than that of original system matrix A, the order of polynomial $\det(sE - \overline{A})$ must be less than that of polynomial $\det(sE - A)$. However if equation(5.143) is found, there must be the phenomenon of pole zero cancellation in equation(5.141). Hence, the hypothesis that system $\sum(A, \quad B, \quad C)$ is not a minimum realization is not found, that's to say, system $\sum(A, \quad B, \quad C)$ is a minimum realization of transfer function $G(s)$.

Necessary demonstration.

If there is no pole zero cancellation in equation(5.141), system $\sum(A, \quad B, \quad C)$ must be controllable and observable. If pole zero cancellation exists in equation(5.141), expression $C(sE - A)^{-1}B$ will be degenerated into a deflation transfer function. According to this deflation transfer function which has no pole zero cancellation, we can find a smaller dimension realization. On the contrary, if there is no pole zero cancellation in equation(5.141), expression $C(sE - A)^{-1}B$ must be a minimum realization, namely system $\sum(A, \quad B, \quad C)$ must be controllable and observable.

Hence, we can utilize whether pole zero cancellation exists in system transfer function to judge whether the relative realization is controllable and observable. However, if there is pole zero cancellation in system transfer function, we will not confirm that system is controllable or observable.

5.10 Stabilizability and Detectability

A linear system is said to have stabilzabilityi if all of its unstable modes, if any, are controllable.

If a system is stable, we say that this system has stabilizability. If a system is completely controllable, it is stabilizable. In the general case, the subsystem defined by the modes or states must be stable in order that the system should be stabilizable. For a linear and constant system, the requirement is that all eigenvalues must be in the stable region, that's to say, the left-half of the s -plane for continuous-time system or unit circle of the z -plane for discrete system. If an uncontrollable system has an positive eigenvalue, this system will not be stabilizable.The significance of stabilizability is that even though certain modes can not be controlled by choice of input or feedback, if they are stable(better yet asymtotically stable), these modes will stay bounded(better yet decay to zero). This modal behavior can often be tolerated in the overall control system.

A linear system is said to be detectable if all of its unstable modes, if any, are observable.

If the system is stable, it is detectable. If it is observable, it is also detectable. In general, the condition is met if the subsystem described by modes are stable. In the linear constant case, the requirement is that all eigenvalues fall in the stable region of the complex plane. If the unobservable state is unstable, such system is not detectable. The significance of the detectability property is that if certain modes are unstable and hence subject to growth without bound, at least this undesirable behavior will be obvious from the output signals. No"hidden modes" can be allowed to grow secretly in an unstable fashion.

Chapter summary

This chapter mainly narrates system controllability and observability for linear system. Furthermore, we also analyze the diagonal canonical form of state space expression and the dual relationship between controllability and observability. Besides these content, we also provide the knowledge of controllable and observable standardization for state space expression, canonical forms and their corresponding realization. Finally, we depict the stabilizability and detectability of system. In this chapter, readers need to mainly understand and grasp the relative topic of system cotnrollability and observability such as construct the judgment matrix of controllability and observability to judge the present state of system.

Related Readings

[1]Wang Wanliang. *"Modern Control Engineering"*, Beijing: Higher Education Press, 2011, China.

[2]P.N.Paraskevopoulos. *"Modern Control Engineering"*, New York: CRC Pressr, 2002, USA.

[3]Liu Bao, Tang Wangsheng. *"Modern Control Theory(second edition)"*, Beijing: Machine Industry Press, 2005, China.

[4]John J. D'Azzo, Constantine H. Houpis and Stuart N. Sheldon. *"Linear Control System Analysis and Design with MATLAB"*, New York: Marcel Dekker, 2003, USA.

[5]Robert L. Williams II and Douglas A. Lawrence. *"Linear State-Space Control Systems"*, Hoboken: John Wiley&Sons Ltd, 2007, USA.

[6]Pia L. Kempker, André C.M. Ran, Jan H. van Schuppen. *"Controllability and observability of coordinated linear systems"*, Linear Algebra and its Applications, Vol437, Iss1, 2012, pp121-pp167.

[7]J.L. Willems. *"Structural controllability and observability"*, Systems & Control Letters, Vol 8, Iss1,1986, pp5-pp12.

[8]Vasile Lupulescu, Awais Younus. *"On controllability and observability for a class of linear impulsive dynamic systems on time scales"*, Mathematical and Computer Modelling, Vol,54, Iss. 5–6, 2011, pp1300-pp1310.

[9]Biswa Nath Datta. "Numerical Methods for Linear Control Systems", Pittsburgh: Academic Press, 2003, USA.

Review Questions

5.1. Please narrate the sufficient and necessary condition of system controllability.

5.2. How is the controllability of system output judged?.

5.3. Please depict how to transform a standard state space into a diagonal canonical form.

5.4. What case is the Jordan matrix suitable for?

5.5. How is the system transfer function transformed into the first controllable canonical form?

5.6.Please narrate the main idea of canonical decomposition for system controllability.

5.7.Please demonstrate the whole deduction course of diagonal canonical form.

5.8.Please repeat the dual principle of state space expression.

5.9.Please explain the specification decomposition of linear time-constant system.

5.10.How could a minimum realization of given system be found?

5.11.How do we judge a given realization is a minimum realization of a given control system?

5.12.How do we judge output controllability of a given control system?

5.13.How do we judge observability of a given control system?

Problems

Problem5.1.Is the following system completely controllable and completely observable?

$$\dot{x}=\begin{bmatrix}-1 & 2 & -1\\ 0 & -3 & 0\\ 1 & -2 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u,\quad y=\begin{bmatrix}4 & 2 & 1\end{bmatrix}x$$

Problem5.2.Please try to judge the state controllability and output observability of the following system:

$$\dot{x}=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 1 & 2\end{bmatrix}x+\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}u\quad y=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}x$$

Problem5.3.Please try to transform the following system into controllable canonical form.

$$\dot{x}=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}x+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

Problem5.4.Please confirm parameters a, b in the following state space expression.

$$\dot{x}=\begin{bmatrix}a & 2\\ 0 & b\end{bmatrix}x+\begin{bmatrix}1\\ 1\end{bmatrix}u,\quad y=\begin{bmatrix}1 & -1\end{bmatrix}x$$

Problem5.5.Please acquire the minimum realization of following transfer function matrix.

$$G(s)=\begin{bmatrix}\frac{s+2}{s+5} & \frac{s}{s+2}\\ \frac{s}{s+2} & \frac{s+1}{s+3}\end{bmatrix}$$

Problem5.6.Please acquire the observable canonical form on the following system.

$$G(s)=\frac{s+2}{s^3+2s^2+3s+6}$$

Problem5.7.Please investigate the controllability and observability of the following system.

$$A=\begin{bmatrix}-6 & 1 & 0\\ -11 & 0 & 1\\ -6 & 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}1\\ 6\\ 5\end{bmatrix},\quad C=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

Problem5.8.Please find the minimum realization of the following system.

$$\dot{x}=\begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 2 & 3 & 1\\ 0 & 0 & 3 & 0\\ 1 & 1 & 2 & 4\end{bmatrix}x+\begin{bmatrix}1\\ 2\\ 0\\ 2\end{bmatrix}u,\quad y=\begin{bmatrix}2 & 3 & -1 & -2\end{bmatrix}x$$