Optimal Control

This chapter provides optimal control theory for readers.In this chapter， the optimal control conception and variation method are respectively narrated and explained to help readers to understand optimal control problem.Furthermore maximum principle and linear quadratic optimal control and dynamic programming will be also narrated and analyzed to aid readers to correctly apply optimal control into practice. In this chapter, readers need to grasp three basic methods: variation method, minimum principle and dynamic programming.

Objectives

When you have learned this chapter, you should be able to:

- Know what optimal control is.
- Understand basic methods of realizing optimal control of control system.
- Grasp core factors of minimum principle and dynamic programming.
- Recognize Bellman equation.

7.1 Concept of Optimal Control

In chapter6, as we have known, if system is completely controllable, a state feedback matrix is always designed to make that closed-loop poles are equal to the desired poles to reach the dynamic requirement desired. Control system indexes designed in terms of pole assignment is mainly determined by the selection of the desired poles of closed-loop control system; however in practical system, some indexes are maximum or optimal in control system. Hence, optimal control is also a kind of design method which core problem is how to select control signal to make system indexes be optimal in some significance.

In engineering, people always wish to find a best scheme to acquire optimal effect. Similar engineering includes optimal problem.

For example, there are two warehouses where some goods are placed. There are 2000 merchandises in warehouse A, and 1500 ones in warehouse B.Three workshops I, II and III respectively need 900, 600 and 1000 merchandises. If the merchandises in warehouse A are carried respectively to the three workshops I, II and III, the relative service fee is 1 $, 2$ and 4$. If the merchandises in warehouse B are carried apart to the three workshops I, II and III, the service fee is 3$, 5$ and 8$. How is these merchandises carried to make that transportation cost become cheapest? In nature, this is an optimal allocation problem.

We postulate that the number of merchandise from warehouse A to workshop I, II and III is respectively x_1, x_2, x_3 ; and that the number of merchandise from warehouse B to workshop I, II and III is respectively x_4, x_5, x_6. Then the total fee could be expressed as

$$f(x) = x_1 + 2x_2 + 4x_3 + 3x_4 + 5x_5 + 8x_6$$

In optimal problem, function $f(x)$ is called aim function. The task of optimization is to confirm variable x to make that function $f(x)$ get optimal value such as minimum value or maximum value. But variable x is not free but limited by objective conditions such as

$$x_1 + x_2 + x_3 \le 1200, \quad x_4 + x_5 + x_6 \le 1800$$

$$x_1 + x_4 = 900, \quad x_2 + x_5 = 600, \quad x_3 + x_6 = 1000$$

In this example, if aim function and constraint condition is a one-power function of variable x, such optimal problem is called linear optimization problem. For the serving fee from warehouse A to workshop I, II and III is cheaper than that from warehouse B to workshop I, II and III, we should firstly select merchandises from warehouse A. So that the above inequality could be converted into the following equality.

$$x_1 + x_2 + x_3 = 1200, \quad x_4 + x_5 + x_6 = 1800$$

Then above example will become an optimization problem with equality constraint.

Aim function and constraint conditions are not only limited to linear cases, the variable of aim function is possibly nonlinear. Hence aim function J is generally expressed as

$$J(x) = f(x) \tag{7.1}$$

Equality constraint condition:

$$g_i\left(\dot{x}\right) = 0, \quad i = 1,2,3,\cdots,m \tag{7.2}$$

Inequality constraint condition:

$$h_j(x) \le 0, \quad j = 1,2,3,\cdots,l \tag{7.3}$$

The optimal task is to find suitable variable x to make aim function be the most optimal in all possible value.

If variable is constant or not related to time, such optimal problem is called static optimization problem. Otherwise, optimal problem is called dynamic optimization problem.

In practice, there are all kinds of optimal problem, and we do not possibly discuss all cases. In this chapter, we only discuss the optimal problem which could be depicted by system state equation.

The prerequisite of optimal control is narrated as follows:

1. State equation which is used to depict control system dynamically.

A given system state equation is

$$\dot{x} = f(x, \quad u, \quad t) \tag{7.4}$$

Here, function f is n dimension vector function, and it can be continuously derived for state variable x and time variable t.

For discrete-time control system, its state equation can be expressed as

$$x(k+1) = f[x(k), \quad u(k), \quad k] \tag{7.5}$$

2.Definite control scope

In actual engineering, control vector $u(t)$ can not be usually any value in given space R. For example, control voltage and control power in system can not be randomly large, which have to be limited by some constraint conditions.

$$\varphi_j(x, \quad u) \le 0, \quad j = 1,2,\cdots,m(m \le r) \tag{7.6}$$

The set of all points which can satisfy above expressed is marked as

$$U = \{u(t)|_{\varphi_j(x, \quad u) \le 0}\} \tag{7.7}$$

Equation(7.7) is said to control set. Control law $u(t)$ which belongs to set $u(t) \in U$ is called admissible control.

3.Definite initial condition

Usually, the initial time of control system is given and definite. If initial state $x(t_0)$ is given, such state is called fixed end. If initial state $x(t_0)$ is random, such state is called free end. If initial state satisfies some constraint condition:

$$\rho_j[x(t_0)] = 0, \quad j = 1,2,\cdots,m(m \le n) \tag{7.8}$$

the relative initial end set is as follows:

$$\Omega_0 = \{x(t_0)|_{\rho_j[x(t_0)]=0}\} \tag{7.9}$$

Then, $x(t_0) \in \Omega_0$, such initial condition is said to variable end.

4.Definite terminal condition

Similar to initial condition, fixed end means that end time t_f and end state $x(t_f)$ are given. Free end means that state $x(t_f)$ may be random under the condition of given time t_f. Variable end means such case $x(t_f) \in \Omega_f$. The objective set is formed by constraint condition $\varphi_j[x(t_f)] = 0$.

$$\Omega_f = \left\{x(t_f)|_{\varphi_j[x(t_f)]=0}\right\} \tag{7.10}$$

5.Performance index

Performance index expression could be generally expressed as

$$J = \theta[x(t_f), \quad t_f] + \int_{t_0}^{t_f} L[x(t), \quad u(t), \quad t]dt \tag{7.11}$$

Here, first term is scalar term which is related to final time t_f and final state $x(t_f)$; and this term shows the certain requirement for final state or final time. Hence we term expression $\theta[x(t_f), \quad t_f]$ final term. The second term is integral term demonstrates the requirement satisfied by state vector and control vector in the whole control course. Final hour t_f may be fixed or free, which is up to specific control problem.

For discrete control system, equation(7.11) could be transformed into the following form:

$$J = \theta[x(N)] + \sum_{k=k_0}^{N-1} L[x(k),\ u(k),\ k] \tag{7.12}$$

In equation(7.11) and equation(7.12), the second term is actually dynamic index function. If we do not consider

If final term $\theta[x(t_f),\ t_f]$ is not considered, the corresponding performance index expression can become

$$J = \int_{t_0}^{t_f} L[x(t),\ u(t),\ t]dt \tag{7.13a}$$

$$J = \sum_{k=k_0}^{N-1} L[x(k),\ u(k),\ k] \tag{7.13b}$$

Such index form in equation(7.13) is called integral form of optimal control.

If dynamic index term is not considered, performance index function will be

$$J = \theta[x(t_f),\ t_f] \tag{7.14a}$$

$$J = \theta[x(N),\ N] \tag{7.14b}$$

Equation(7.14) is called final form of optimal control.

Optimal control problem is to find a best control vector $u(t)$ in given control variable collection to make that the performance index of controlled system could acquire optimal index from original state to aim state. Control vector $u(t)$ which could make control system holds best performance indexes is called optimal control $u^*(t)$. The solution of state equation in optimal control vector $u^*(t)$ is called optimal trajectory $x^*(t)$. The performance index J acquired along the optimal trajectory $x^*(t)$ is called optimal index J^*.

If control system is designed in terms of linear quadratic performance indexes, linear control law can be easily realized to acquire success in engineering. The common form of linear quadratic performance index is the following:

$$J = \frac{1}{2}x^T(t_f)Q_0x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Q_1x(t) + u^T(t)Q_2u(t)\right]dt \tag{7.15}$$

Linear quadratic performance index of discrete control system:

$$J = \frac{1}{2}x^T(N)Q_0(N)x(N) + \frac{1}{2}\sum_{k=k_0}^{N-1}\left[x^T(k)Q_1(k)x(k) + u^T(k)Q_2(k)u(k)\right] \tag{7.16}$$

Here, matrices $Q_0(t_f)$, $Q_1(t)$, $Q_2(t)$ and $Q_0(N)$, $Q_1(k)$, $Q_2(k)$ are both weighting matrix.

Definition of Optimal Control Problem *From all among all admissible control functions* $u \in U$, *find that one which minimizes* J *of equation(7.5) subject to the* *dynamic system constraints of equation(7.4) and all initial and terminal boundary conditions that may be specified.*

If the control is determined as a function of the initial state and other given system parameters, such control system is said to be open-loop. If the control is determined as a function of the present state, then such control is a closed-loop or feedback control law.

If the control system is completely controllable, there is at least one control which will transfer any initial state to any desired final state. If control system is not controllable, it is not meaningful to search for the optimal control.

However controllability does not guarantee that a solution exists for every optimal control problem. Whenever the admissible controls are restricted to the set U , certain final states may not be attainable. Even though the system is completely controllable, the required control may not belong to set U .

7.2 Static optimal control

The objective function of static optimal control problem is a multiple variable function in nature. Its optimal solution could be gained by classical differentiation. The objective function of dynamic optimal problem is a functional which extreme value can be acquired by variation method. Now let's review mathematical model of linear programming together.

7.2.1 Mathematical model of linear programming

A company produces two kinds of products A and B . The resource, material and laboring day are listed in the following table7.1; please find how to make maximum profit.

Table7.1 Information of products

Materials	A Product	B Product	Sources
Brass(ton)	9	4	360
Power(Kilowatt)	4	5	200
Working days	3	10	300
Profits(Million RMB per kilogram)	7	12	

Solution:

We postulate that this company may produce product A with x_1 kilogram and product B with x_2 kilogram; and then we can know the total profit expression:

$$S = 7x_1 + 12x_2$$

The mathematical model of this problem can be depicted as the following:

Objective function:

$$\max S = 7x_1 + 12x_2$$

Constraint conditions:

$$\begin{cases} 9x_1 + 4x_2 \le 360 \\ 4x_1 + 5x_2 \le 200 \\ 3x_1 + 10x_2 \le 300 \\ x_i \ge 0 (i = 1, 2, \cdots) \end{cases}$$

Summarize above optimal problem, we can gain the following characteristics:

(1) Each problem is to find the value of a group of variable. The value of some group of variable represents a specific scheme.

(2) There are some constraint conditions. These constraint condition could be expressed by equality or inequality.

(3) A desired object function need to be calculated to find an optimal value.

The optimal problem with above three characteristics comes down to linear programming problem, which mathematical model is said to the mathematical model of linear programming.

The general mathematical form of linear programming model is

$$\max(or \min) f(x) = c_1x_1 + c_2x_2 + c_3x_3 + \cdots\cdots + c_nx_n \tag{7.17}$$

General constraint conditions:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_1 + \cdots + a_{1n}x_n \le (=, \quad \ge) b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_1 + \cdots + a_{2n}x_n \le (=, \quad \ge) b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_1 + \cdots + a_{mn}x_n \le (=, \quad \ge) b_m \\ x_j \ge 0 (j = 1, 2, \cdots, n) \end{cases} \tag{7.18}$$

Variables $x_1, x_2, \cdots, x_n$ are called to decision variable.

The basic steps of building the mathematical model of linear programming:

Step1, confirm decision variables $x_1, x_2, \cdots, x_n$. Decision variables is the base of building mathematical model.

Step2, give out constraint conditions, and add some auxiliary non-negative conditions according to practical problem. Constraint conditions could be expressed by inequality or equality. Usually, table is employed to confirm all limited data to avoid the unnecessary limitation or requirements.

Step3, ensure objective function including ascertaining maximum or minimum. Objective function has to be expressed by linear function of decision variable.

7.2.2 Extreme value of single-variable function

If we postulate that expression $J = f(x)$ is a continuously derived function with single variable in the range $[a, \quad b]$, then necessary condition of finding its extreme value x^* is the following:

$$f'(x)\big|_{x=x^*}=0 \tag{7.19}$$

The necessary and sufficient condition that variable x^* is minimal value is

$$f'(x)\big|_{x=x^*}=0\,,\quad f''(x)>0 \tag{7.20}$$

The necessary and sufficient condition that variable x^* is maximal value is

$$f'(x)\big|_{x=x^*}=0\,,\quad f''(x)<0 \tag{7.21}$$

Extreme point x^* in terms of derivation of objective function is a stationary point, which characteristics are as follows:

(1)When $f''(x)<0$, extreme point x^* is maximal point;

(2)When $f''(x)=0$, extreme point x^* is inflection point;

(3)When $f''(x)>0$, extreme point x^* is minimal point.

These extreme value $f(x^*)$ is only relative to adjacent value $f(x)$; hence they have partial characteristic, which is called to relative extreme value. In practice, there are a lot of extreme points, and then we need to find a minimum among all minimal value $f(x)$. Minimum has global characteristic and it is unique. Minimum is generally marked as

$$J^*=f(x^*)=\min_{x\in X}[f(x)] \tag{7.22}$$

Example7.1 Please find the minimum of the following function which range of variable is $[-1,\ \ 5]$.

$$f(x)=x^4-6x^2+8$$

Solution:

The first order derivative of given function is as follows:

$$f'(x)=4x^3-12x$$

Order $f'(x)=0$, and then gain:

$$x_1=0\,,\quad x_2=\sqrt{3}\,,\quad x_3=-\sqrt{3}\text{ (discard)}$$

Hence, the relative extreme value $f(x)$ can be calculated when $x=-1,\ \ 0,\ \ \sqrt{3},\ \ 5$:

$$f(-1)=3\,,\quad f(0)=8\,,\quad f(\sqrt{3})=-1\,,\quad f(5)=483$$

The second order derivative of given function is

$$f''(x)=12x^2$$

For expression $f''(x)$ is always greater than zero, all points in definition domain are minimal except $x=0$.

Comparing above all extreme values, we can get minimum.

$$J^*=\min\left[f(-1),\ \ f(0),\ \ f(\sqrt{3}),\ \ f(5)\right]=-1$$

Namely, when $x=\sqrt{3}$, minimum $f(\sqrt{3})=-1$.

7.2.3 Extreme value of multiple-variable function

Assume that a general form of multiple-variable function is as follows:

$$f = f(x_1, \quad x_2, \quad \cdots, \quad x_n) \tag{7.23}$$

The condition that above function ascertains extreme value is

$$\frac{\partial f}{\partial x} = 0 \tag{7.24}$$

Or its gradient vector is zero, namely

$$\nabla f_u = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^T = 0 \tag{7.25}$$

The necessary and sufficient condition of minimum is

$$\frac{\partial^2 f}{\partial x^2} > 0 \tag{7.26}$$

Namely, the following positive matrix:

$$\nabla^2 f_x = \frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \tag{7.27}$$

Example7.2 Function $f(x) = x_1^2 + 5x_2^2 + 4x_3^2 + 4x_2x_3 + 4x_1x_3 - 6x_2 + 3$; please find the extreme value points of given function and minimum.

Solution:

In terms of equation(7.24) and equation(7.25), we can acquire

$$\frac{\partial f}{\partial x_1} = 2x_1 + 4x_3 = 0\,; \quad \frac{\partial f}{\partial x_2} = 10x_2 + 4x_3 - 6 = 0\,; \quad \frac{\partial f}{\partial x_3} = 8x_3 + 4x_2 = 0$$

Solve above equation group and we can gain:

$$x_1 = \frac{3}{4}, \qquad x_2 = \frac{3}{4}, \qquad x_3 = -\frac{3}{8}$$

Hence, extreme value point is

$$x = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & -\frac{3}{8} \end{bmatrix}^T$$

The function matrix of second order derivative is:

$$\nabla^2 f_x = \frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 10 & 4 \\ 0 & 4 & 8 \end{bmatrix} > 0$$

Hence, above matrix is positive; and point $x = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & -\frac{3}{8} \end{bmatrix}^T$ is minimum point.

Minimum $f^*(x)$ is the following:

$$f^*(x^*) = \frac{9}{16} + \frac{45}{16} + \frac{36}{64} - \frac{36}{32} - \frac{18}{4} + 3 = \frac{21}{16}$$

7.2.4 Extreme value with the condition of equation constraint

For extreme value with the condition of equation constraint, equivalent transform may convert such problem into extreme value with the condition of no constraint.

Let me give an example, please find the volume of cuboid which volume is maximum and which surface area is a^2.

We can postulate that the radius and height of column is respectively r and h; and then we can get total area expression which is a constraint condition:

$$g(r,\ h) = 2\pi rh + 2\pi r^2 - a^2 = 0 \tag{7.28}$$

And the volume of cuboid is

$$J = V = \pi r^2 h \tag{7.29}$$

The methods of solving such problem have many methods such as substitution method and Lagrange method.

(1) Substitution method

From equation(7.28), the expression of variable h could be expressed as

$$h = \frac{a^2 - 2\pi r^2}{2\pi r} \tag{7.30}$$

Equation(7.30) is substituted into equation(7.29), and then given problem can be transformed into no constraint problem.

$$V = \pi r^2 h = \pi r^2 \left[\frac{a^2 - 2\pi r^2}{2\pi r} \right] = \frac{a^2 r - 2\pi r^3}{2} \tag{7.31}$$

The derivative of equation(7.31) for variable r is as follows:

$$\frac{\partial V}{\partial r} = \frac{a^2}{2} - 3\pi r^2 = 0 \tag{7.32}$$

The solution of equation(7.32) is $r^* = \frac{a}{\sqrt{6\pi}}$, $h^* = a\sqrt{\frac{2}{3\pi}}$.

For $\frac{\partial^2 V}{\partial r^2} = -6\pi r < 0$, above extreme point is maximum. Maximum volume is

$$J^* = V(r^*,\quad h^*) = \frac{a}{6}\sqrt{\frac{2}{3\pi}} \tag{7.33}$$

(2) Lagrange method

According to aim function of equation(7.29) and constraint condition of equation(7.28), we can construct Lagrange function:

$$H = J + \lambda g(r,\quad h) = \pi r^2 h + \lambda\left(2\pi r h + 2\pi r^2 - a^2\right) \tag{7.34}$$

Equation(7.34) is a no constraint function with three variable. Its extreme conditions are as follows:

$$\frac{\partial H}{\partial r} = 2\pi r h + \lambda(2\pi h + 4\pi r) = 0, \quad \frac{\partial H}{\partial h} = \pi r^2 + 2\lambda \pi r = 0, \quad \frac{\partial H}{\partial \lambda} = 2\pi r h + 2\pi r^2 - a^2 = 0$$

Solve above equation group to gain extreme point:

$$r^* = \frac{a}{\sqrt{6\pi}}, \qquad h^* = a\sqrt{\frac{2}{3\pi}}, \qquad \lambda^* = -\frac{a}{2}\sqrt{\frac{1}{6\pi}} \tag{7.35}$$

If extreme points of equation(7.35) is substituted into equation(7.34), we easily find that expression $\lambda g(r,\quad h) = 0$ can be found.

Summarizing above Lagrange function, we can write common case.

Assume a continuous and derivative aim function:

$$J = f(x,\quad u) \tag{7.36}$$

Equality constraint condition is

$$g(x,\quad u) = 0 \tag{7.37}$$

Here, sign x is n-dimension column vector; sign u is r-dimension column vector; and sign g is n-dimension column function.

In terms of Lagrange multiplier, we can write out Lagrange aim function:

$$H = J + \lambda^T g(x,\quad u) = f(x,\quad u) + \lambda^T g(x,\quad u) \tag{7.38}$$

Utilizing multiple variable function to solve extreme value, we can know that the necessary condition of aim function which could have extreme value is the following:

$$\frac{\partial H}{\partial x} = 0, \qquad \frac{\partial H}{\partial u} = 0, \qquad \frac{\partial H}{\partial \lambda} = 0 \tag{7.38}$$

Namely,

$$\frac{\partial f}{\partial x} + \left(\frac{\partial g}{\partial x}\right)^T \lambda = 0 \tag{7.39}$$

$$\frac{\partial H}{\partial u} = \frac{\partial f}{\partial u} + \left(\frac{\partial g}{\partial u}\right)^T \lambda = 0 \tag{7.40}$$

$$\frac{\partial H}{\partial \lambda} = g(x,\quad u) = 0 \tag{7.41}$$

Here,

$$\left(\frac{\partial g}{\partial x}\right)^T=\begin{bmatrix}\frac{\partial g_1}{\partial x}\\ \frac{\partial g_2}{\partial x}\\ \vdots\\ \frac{\partial g_n}{\partial x}\end{bmatrix}^T=\begin{bmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n}\end{bmatrix}_{n\times n} \tag{7.42}$$

Example7.3 Please find the extreme value x^* and u^* of aim function

$$J=f(x,\quad u)=\frac{1}{2}x^TQ_1x+\frac{1}{2}u^TQ_2u\ ;$$

It has to satisfy the constraint condition $g(x,\quad u)=x+Fu+d=0$.

Matrices Q_1 and Q_2 are positive matrices; and matrix F is random matrix.

Solution: Construct Lagrange aim function:

$$H=J+\lambda^Tg(x,\quad u)=\frac{1}{2}x^TQ_1x+\frac{1}{2}u^TQ_2u+\lambda^T(x+Fu+d)$$

The extreme condition of above Lagrange aim function:

$$\frac{\partial H}{\partial x}=Q_1x+\lambda=0\,,\quad \frac{\partial H}{\partial u}=Q_2u+F\lambda=0\,,\quad \frac{\partial H}{\partial \lambda}=x+Fu+d=0$$

Combine and solve the extreme point:

$$u^*=-\left(Q_2+F^TQ_1F\right)^{-1}F^TQ_1d$$

$$x^*=-\left[E-F\left(Q_2+F^TQ_1F\right)^{-1}F^TQ_1\right]d$$

$$\lambda^*=\left[Q_1-Q_1F\left(Q_2+F^TQ_1F\right)^{-1}F^TQ_1\right]d$$

Since matrices Q_1 and Q_2 are positive matrices, obviously extreme point could satisfy the minimum condition.

7.3 Extreme Condition of Variation Method and Functional

7.3.1 Functional concept

For independent variable, if there is a function $\{x(t)\}$, a certain value J is related to each function $x(t)$; then we think that variable J is a functional which depends on function $x(t)$, marked as $J[x(t)]$.

Apparently, the independent variable x of functional $J[x(t)]$ is the function of time variable t . Hence sometimes functional $J[x(t)]$ is called to the function of function. The difference between function and functional is that the independent variable of functional is function and that the argument of function is variable.

If functional $J[x(t)]$ satisfy the following relationship:

$$J[ax(t)] = aJ[x(t)] \tag{7.43a}$$

$$J[x(t)+y(t)] = J[x(t)] + J[y(t)] \tag{7.43b}$$

then functional $J[x(t)]$ is linear functional; in equation(7.43), sign a is real and functions $x(t)$ and $y(t)$ are the function in function space.

7.3.2 Functional variation

The variation δx of functional variable $x(t)$ is defined as

$$\delta x = x(t) - x^*(t) \tag{7.44}$$

Function $x^*(t)$ is aim function an optimal trajectory in optimal control; and function $x(t)$ is some function which have the same function space as aim function $x^*(t)$.

The increment of functional $J[x(t)]$ could be expressed as the following

$$\Delta J[x(t),\quad \delta x] = J[x(t)+\delta x] - J[x(t)] \tag{7.45}$$

If equation(7.45) can be converted into the following form

$$\Delta J[x(t),\quad \delta x] = L[x(t),\quad \delta x] + \beta[x(t),\quad \delta x]\|\delta x\| \tag{7.46}$$

And when $\|\delta x\| \to 0$, $\beta[x(t),\quad \delta x] \to 0$ and $L[x(t),\quad \delta x]$ is linear functional of variation δx, expression $L[x(t),\quad \delta x]$ is called to the one order variation of functional $J[x(t)]$, marked as δJ.

In terms of above definition of variation, we can know that the variation of functional is a kind of linear mapping; hence the calculation rules of variation is similar to linear calculation of function.

Assume that F_1 and F_2 are both the function of function x, derivative $\dot{x}$ and variable t; the following variation rules are always found:

1) $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$

2) $\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$

3) $\delta \int F\left(x,\quad \dot{x},\quad t\right)dt = \int \delta F\left(x,\quad \dot{x},\quad t\right)dt$

4) $\delta \dot{x} = \dfrac{d}{dt}\delta x$

Example7.4 Assume F_1 and F_2 is the derivable and continuous function of variables x, y and y'. Please try to testify above rules of variation rules.

Demonstration:

1) $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$

$$\begin{aligned}\delta(F_1+F_2) &= \frac{\partial}{\partial y}(F_1+F_2)\delta y+\frac{\partial}{\partial y'}(F_1+F_2)\delta y' \\ &= \left(\frac{\partial F_1}{\partial y}\delta y+\frac{\partial F_2}{\partial y}\delta y\right)+\left(\frac{\partial F_1}{\partial y'}\delta y'+\frac{\partial F_2}{\partial y'}\delta y'\right) \\ &= \left(\frac{\partial F_1}{\partial y}\delta y+\frac{\partial F_1}{\partial y'}\delta y'\right)+\left(\frac{\partial F_2}{\partial y}\delta y+\frac{\partial F_2}{\partial y'}\delta y'\right) \\ &= \delta F_1+\delta F_2\end{aligned}$$

2) $\delta(F_1F_2)=F_1\delta F_2+F_2\delta F_1$

$$\begin{aligned}\delta(F_1F_2) &= \frac{\partial}{\partial y}(F_1F_2)\delta y+\frac{\partial}{\partial y'}(F_1F_2)\delta y' \\ &= \left(F_2\frac{\partial F_1}{\partial y}+F_1\frac{\partial F_2}{\partial y}\right)\delta y+\left(F_2\frac{\partial F_1}{\partial y'}+F_1\frac{\partial F_2}{\partial y'}\right)\delta y' \\ &= F_1\left(\frac{\partial F_2}{\partial y}\delta y+\frac{\partial F_2}{\partial y'}\delta y'\right)+F_2\left(\frac{\partial F_1}{\partial y}\delta y+\frac{\partial F_1}{\partial y'}\delta y'\right) \\ &= F_1\delta F_2+F_2\delta F_1\end{aligned}$$

3) $\delta\int F(x,\ y,\ y')dx=\int\delta F(x,\ y,\ y')dx$

$$\begin{aligned}\delta\int F(x,\ y,\ y')dx &= \frac{\partial}{\partial y}\left[\int F(x,\ y,\ y')dx\right]\delta y+\frac{\partial}{\partial y'}\left[\int F(x,\ y,\ y')dx\right]\delta y' \\ &= \int\left[\frac{\partial}{\partial y}F(x,\ y,\ y')\delta y\right]dx+\int\left[\frac{\partial}{\partial y'}F(x,\ y,\ y')\delta y'\right]dx \\ &= \int\left[\frac{\partial}{\partial y}F(x,\ y,\ y')\delta y+\frac{\partial}{\partial y'}F(x,\ y,\ y')\delta y'\right]dx \\ &= \int\delta F(x,\ y,\ y')dx\end{aligned}$$

4) $\delta\dot{y}=\frac{d}{dx}\delta y$

$$\begin{aligned}\delta\dot{y} &= \frac{\partial}{\partial x}\left(\frac{dy}{dx}\right)\delta x+\frac{\partial}{\partial x'}\left(\frac{dy}{dx}\right)\delta x' \\ &= \frac{d}{dx}\left(\frac{\partial y}{\partial x}\right)\delta x+\frac{d}{dx}\left(\frac{\partial y}{\partial x'}\right)\delta x' \\ &= \frac{d}{dx}\left[\left(\frac{\partial y}{\partial x}\right)\delta x+\left(\frac{\partial y}{\partial x'}\right)\delta x'\right] \\ &= \frac{d}{dx}\delta y\end{aligned}$$

7.3.3 Functional extreme

Assume that function $x_0(t)$ is some function in all permissible function set X of functional $J[x(t)]$, for $x \in X$, if the following inequality is always found:

$$J[x(t)] \leq J[x_0(t)], \text{ or } J[x(t)] \geq J[x_0(t)] \quad (7.46)$$

then it's thought that functional $J[x(t)]$ reaches maximum or minimum at $x_0(t)$. And the curve of function $x_0(t)$ is called to the extreme curve of functional $J[x(t)]$.

If functional $J[x(t)]$ reaches extreme value at $x(t) = x_0(t)$, the necessary condition is

$$\delta J\Big|_{x=x_0(t)} = \delta J[x_0(t)] = 0, \text{ or } \delta J(x^*, \quad \Delta x) = \frac{d}{d\varepsilon}\delta J(x^* + \varepsilon \Delta x)\Big|_{\varepsilon=0} = 0 \quad (7.47a)$$

Demonstration:

Assume that the following expression reaches extreme value at $a = 0$.

$$\phi(a) = J[x_0(t) + a\delta x]$$

Hence,

$$\begin{aligned}
\phi'(0) &= \lim_{a\to 0}\frac{\phi(a) - \phi(0)}{a} \\
&= \lim_{a\to 0}\frac{J[x_0(t) + a\delta x] - J[x_0(t)]}{a} \\
&= \lim_{a\to 0}\frac{J[x_0(t) + a\delta x] + \beta(x_0, \quad a\delta x)\|a\delta x\|}{a} \\
&= \lim_{a\to 0}\{J[x_0(t), \quad \delta x] \pm \beta(x_0, \quad a\delta x)\max|\delta x|\}
\end{aligned}$$

When $a \to 0$, $a\delta x \to 0$; then $\beta(x_0, \quad a\delta x) \to 0$, hence

$$\phi'(0) = J[x_0(t), \quad \delta x_0] = \delta J = 0$$

For multiple -variable functional,

$$J = J[x_0(t), \quad x_1(t), \quad \cdots\cdots, \quad x_n(t)] \quad (7.47b)$$

In equation(7.47b), $x_0(t)$, $x_1(t)$, $\cdots$, $x_n(t)$ are both augmented functions.

Variation δJ of multiple-variable function is defined as follows

$$\delta J = J[x_0(t) + a\delta x_0, \quad x_1(t) + a\delta x_1, \quad \cdots\cdots, \quad x_n(t) + a\delta x_n] \quad (7.47c)$$

The necessary condition that multiple-variable functional gets extreme is as follows:

$$\delta J = 0 \quad (7.47d)$$

In practice, the optimal trajectory curve $x^*(t)$ of functional $J[x(t)]$ is not usually selected at random, but limited by all kinds of constraints. In order to explore the extremum problem of functional $J[x(t)]$, let's discuss the extreme problem of the following simplest functional:

$$J[x(t)] = \int_{t_0}^{t_1} F\left(x, \quad \dot{x}, \quad t\right)dt \quad (7.48)$$

And equation(7.48) has to satisfy the boundary condition:

$$x(t_0)=x_0\,,\qquad x(t_1)=x_1 \tag{7.49}$$

Now let's explore the condition when trajectory curve $x(t)$ reaches maximum or minimum. Assume that all permissible curves could be expressed by optimal curve $x(t)$ and real coefficient a and the variation of curve $x(t)$, namely:

$$\bar{x}(t)=x(t)+a\delta x \tag{7.50}$$

Term δx is the variation of function $x(t)$, namely

$$\delta x\big|_{t=t_0}=\delta x\big|_{t=t_1}=0 \tag{7.51}$$

Obviously, $\bar{x}\big|_{t=t_0}=x_0$, $\bar{x}\big|_{t=t_1}=x_1$. When $a=0$, $\bar{x}=x(t)$ is extreme curve when functional $J[x(t)]$ reaches extreme. Expression(7.50) is substituted into equation(7.48), and then we can get the following equation:

$$\phi(a)=J[x(t)+a\delta x]=\int_{t_0}^{t} F(t,\quad x+a\delta x,\quad x'+a\delta x')dt \tag{7.52}$$

Since $\phi(a)$ gets extreme at $a=0$, in terms of the necessary condition of extreme, the following expression always exists:

$$\phi'(0)=\frac{\partial}{\partial a}J[x(t)+a\delta x]\bigg|_{a=0}=\int_{t_0}^{t_1}\left(\frac{\partial F}{\partial x}\delta x+\frac{\partial F}{\partial x'}\delta x'\right)dt=0 \tag{7.53}$$

According to integration by parts and equation(7.51), there is the following expression

$$\int_{t_0}^{t_1}\frac{\partial F}{\partial x'}\delta x'dt=\int_{t_0}^{t_1}\frac{\partial F}{\partial x'}d(\delta x)=\left[\frac{\partial F}{\partial x'}\delta x\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\delta x\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)dt=-\int_{t_0}^{t_1}\delta x\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)dt \tag{7.54}$$

Equation(7.54) is replaced into equation(7.53), and then yields to

$$\phi'(0)=\delta J=\int_{t_0}^{t_1}\left[\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\right]\delta x dt \tag{7.55}$$

If equation(7.55) is always found, then the following equation has to exist:

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)=0 \tag{7.56a}$$

Namely, If functional $J[x(t)]$ gets extreme, the function $x(t)$ of functional $J[x(t)]$ has to satisfy the above differential equation. The necessary condition that functional $J[x(t)]$ gets extreme is the following:

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)=0\,,\quad \left[\frac{\partial F}{\partial \dot{x}}\delta x\right]_{t_0}^{t_f}=0 \tag{7.56b}$$

Equation(7.56) is called to the Euler equation of functional $J[x(t)]$.

Example7.5 Please find the optimal curve of functional $J=J[x(t)]=\int_0^1(x'^2+10tx)dt$. Its constraint conditions are $x(0)=0$ and $x(1)=1$.

Solution:

From given functional $J[x(t)]$, we can know:

$$F = (x')^2 + 10tx$$

According to Euler equation(7.56), we can write out the Euler equation of given functional:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 10t - 2x'' = 0$$

Solve above differential equation, and we can get

$$x(t) = \frac{5}{6}t^3 + C_1 t + C_2$$

Given constraint condition is $x(0) = 0$ and $x(1) = 1$, hence the unknown coefficient of indefinite integral $x(t)$ can be solved:

$$C_1 = \frac{1}{6} \qquad C_2 = 0$$

Hence the extreme curve $x^*(t)$ of functional $J[x(t)]$ is

$$x^*(t) = \frac{5}{6}t^3 + t$$

The extreme problem under equality is called the extreme problem of condition functional $J[x(t)]$. For such problem, it is usually transformed into the extreme problem of no constraint condition by Lagrange multiplier in equation(7.38). For the form of functional $J[x(t)]$ in equation(7.48), Euler equation in equation(7.56) can be utilized, too.

7.4 Optimal Control Solution of Unconstrained System Using Variation Method

We postulate that state equation of system is as follows:

$$\dot{x} = f[x(t), \quad u(t), \quad t], \qquad x(t_0) = x_0 \tag{7.57a}$$

Performance index expression is

$$J = \theta[x(t_f), \quad t_f] + \int_{t_0}^{t_f} L[x(t), \quad u(t), \quad t]dt \tag{7.57b}$$

θ and L are scalar function.

The optimal control problem is to confirm the necessary condition of functional $J[x(t)]$ reaching extreme value under the constraint condition of state equation in equation(7.57). If control variable $u(t)$ is not limited, such control problem will become no constraint control problem.

If final term $\theta[x(t_f), \quad t_f]$ is not considered, the corresponding performance index expression may become the form of equation(7.48).

7.4.1 Optimal control solution by Euler equation group

If the final term of performance index expression is not considered, it will become

$$J=\int_{t_0}^{t_f} L[x(t),\quad u(t),\quad t]dt \tag{7.58}$$

Although the form of equation(7.58) is the same as that of equation(7.48), they are different in nature. Equation(7.58) is a multiple-variable functional which includes two variables $x(t)$ and $u(t)$. Adjust the form of equation(7.58) into the following:

$$J[x(t),\quad u(t)]=\int_{t_0}^{t_1} L[t,\quad x(t),\quad u(t),\quad x'(t),\quad u'(t)]dt \tag{7.59}$$

The boundary condition of equation(7.59) is such

$$\begin{cases} x(t_0)=x_0 \\ u(t_0)=u_0 \end{cases},\qquad \begin{cases} x(t_1)=x_1 \\ u(t_1)=u_1 \end{cases} \tag{7.60}$$

In equation(7.59), the aim function integrated is a five-variable continuous function; and the functions $x(t)$ and $u(t)$ have second-order continuous partial derivatives. The discussing method employed here is similar to that in section7.3.3. For real α and β, we can construct function sets as the following:

$$\bar{x}=x(t)+\alpha\delta x,\qquad \bar{u}=u(t)+\beta\delta u \tag{7.61}$$

Equation(7.62) is the adjacent curve of the following equation:

$$\begin{cases} x=x(t) \\ u=u(t) \end{cases} \tag{7.62}$$

δx and δu are the variation of variables x and y; namely,

$$\delta x|_{t_0}=\delta u|_{t_0}=\delta x|_{t_1}=\delta u|_{t_1}=0 \tag{7.63}$$

Equation(7.61) is substituted into equation(7.59), and then we can get

$$\begin{aligned} J[\bar{x},\quad \bar{u}] &= J[x(t)+\alpha\delta x,\quad u(t)+\beta\delta u] \\ &= \int_{t_0}^{t_1} L(t,\quad x+\alpha\delta x,\quad u+\beta\delta u,\quad x'+\alpha\delta x',\quad u'+\beta\delta u')dt \end{aligned} \tag{7.64}$$

Equation(7.64) gets extreme at $\alpha=\beta=0$, hence

$$\left.\frac{\partial J}{\partial\alpha}\right|_{\alpha=\beta=0}=0,\qquad \left.\frac{\partial J}{\partial\beta}\right|_{\alpha=\beta=0}=0 \tag{7.65}$$

Namely,

$$\left.\frac{\partial J}{\partial\alpha}\right|_{\alpha=\beta=0}=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\delta x+\frac{\partial L}{\partial x'}\delta x'\right)dt=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right)\right]\delta x dt=0 \tag{7.66a}$$

$$\left.\frac{\partial J}{\partial\beta}\right|_{\alpha=\beta=0}=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial u}\delta u+\frac{\partial L}{\partial u'}\delta u'\right)dt=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial u}-\frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right)\right]\delta u dt=0 \tag{7.66b}$$

So that the necessary condition that functional $J[x(t),\quad u(t)]$ gets extreme is such:

$$\begin{cases} \dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial x'}\right) = 0 \\ \dfrac{\partial L}{\partial u} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial u'}\right) = 0 \end{cases} \tag{7.67}$$

Equation group(7.67) is called to Euler equation group of equation(7.59).

Example7.5 Please find the optimal trajectory curve of functional $J[x(t),\quad u(t)] = \int_0^1 \left(x'^2 + u'^2 + 2xu\right)dt$,which satisfy the following constraint conditions.

$$\begin{cases} x(0) = 0, \quad x\left(\dfrac{\pi}{2}\right) = 1 \\ u(0) = 0, \quad u\left(\dfrac{\pi}{2}\right) = -1 \end{cases}$$

Solution:

$L = x'^2 + u'^2 + 2xu$, hence Euler equation group is the following:

$$\begin{cases} 2u - \dfrac{d}{dt}\left(2x'\right) = 0 \\ 2x - \dfrac{d}{dt}\left(2u'\right) = 0 \end{cases}$$

Namely,

$$\begin{cases} u - x'' = 0 \\ x - u'' = 0 \end{cases}$$

Eliminate the variable u to yield to

$$x - x^{(4)} = 0$$

Its general solutions of above differential equation are

$$\begin{cases} x = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t \\ u = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t \end{cases}$$

According to given constraint condition, we can get the unknown number of above equation group.

$$C_1 = C_2 = C_3 = 0, \quad C_4 = 1$$

Hence, optimal trajectory curve is

$$\begin{cases} x = \sin t \\ u = -\sin t \end{cases}$$

~~**Example7.6**~~ Please find the optimal trajectory curve $x^*(t)$and ~~control law~~ u^*~~(t)~~ of the following functional $J = \int_0^3 \frac{1}{2}\left[u^2(t) - u'^2(t)\right]dt$,which satisfy the following constraint conditions.

$$\dot{x} = -u\,, \quad u(0) = 0\,, \quad u(3) = 6\,, \quad x(0) = 0$$

Solution

$L = \frac{1}{2}\left[u^2(t) - u'^2(t)\right]$, in terms of equation(7.67), we can get the following Euler equation:

$$u - u'' = 0$$

Solve above differential equation to get the following solution:

$$u(t) = C_1 e^t + C_2 e^{-t}$$

In terms of given constraint condition, we may get the unknown number:

$$C_1 = \frac{6e^3}{e^6 - 1}, \qquad C_2 = -\frac{6e^3}{e^6 - 1}$$

Hence, optimal control law u^* is the following:

$$u^* = \frac{6e^3}{e^6 - 1} e^t - \frac{6e^3}{e^6 - 1} e^{-t}$$

Optimal trajectory curve x^*:

$$x^* = -\frac{6e^3}{e^6 - 1} e^t - \frac{6e^3}{e^6 - 1} e^{-t} + \frac{12e^3}{e^6 - 1}$$

7.4.2 Optimal control solution by Lagrange multiplier

If equation(7.57a) is the constraint condition of equation(7.57b), we may utilize Lagrange extreme method to gain the extreme point. Specific deducing course is as follows.

Constructing augmented functional J_a is

$$J_a = \theta\left[x(t_f),\ t_f\right] + \int_{t_0}^{t_f} \left\{ L[x(t),\ u(t),\ t] + \lambda^T \left[f(x,\ u,\ t) - \dot{x} \right] \right\} dt \tag{7.68}$$

Constructing Hamilton function is

$$H(x,\ u,\ \lambda,\ t) = L(x,\ u,\ t) + \lambda^T f(x,\ u,\ t) \tag{7.69}$$

Here, $\lambda \in R^n$ is Lagrange multiplier vector, then augmented functional is

$$J_a = \theta\left[x(t_f),\ t_f\right] + \int_{t_0}^{t_f} \left[H(x,\ u,\ \lambda,\ t) - \lambda^T \dot{x} \right] dt \tag{7.70}$$

We postulate that the original time t_0 and its state are both given, $x(t_0) = x_0$. In terms of final boundary condition, the following cases will be discussed in the following:

Case1, final time t_f is given and random, namely, final state $x(t_f)$ is random.

Augmented functional J_a is

$$J_a=\theta\left[x(t_f)\right]+\int_{t_0}^{t_f}\left[H(x,\quad u,\quad \lambda,\quad t)-\lambda^T\dot{x}\right]dt \tag{7.71}$$

One-order variation of augmented functional J_a is taken and ordered to be zero:

$$\delta J_a=\left(\frac{\partial\theta}{\partial x}\right)^T\delta x+\int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}\right)^T\delta\lambda-\dot{x}^T\delta\lambda-\lambda^T\delta\dot{x}\right]dt=0 \tag{7.72}$$

In terms of partial integral method, the following expression can be known:

$$\int_{t_0}^{t_f}\lambda^T\delta\dot{x}dt=\int_{t_0}^{t_f}\lambda^T d(\delta x)=\left[\lambda^T\delta x\right]_{t_0}^{t_f}-\int_{t_0}^{t_f}\dot{\lambda}^T\delta x dt \tag{7.73}$$

Equation(7.73) is replaced into equation(7.72) and expression $\delta x(t_0)=0$ should be noted. Then we can get

$$\delta J_a=\left(\frac{\partial\theta}{\partial x}-\lambda\right)^T\delta x\Bigg|_{t=t_f}+\int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}-\dot{x}\right)^T\delta\lambda\right]dt=0 \tag{7.74}$$

Since above variations δx, δu and $\delta\lambda$ are both random, one-order variation δJ_a of augmented performance index functional J_a is zero, namely the necessary condition of that equation(7.57) gets extreme is as follows:

1) Positive equation

State equation: $$\dot{x}=\frac{\partial H(x,\quad u,\quad \lambda,\quad t)}{\partial\lambda} \tag{7.75}$$

Adjoining(costate) equation: $$\dot{\lambda}=-\frac{\partial H(x,\quad u,\quad \lambda,\quad t)}{\partial x} \tag{7.76}$$

2) Control equation: $$\frac{\partial H(x,\quad u,\quad \lambda,\quad t)}{\partial u}=0 \tag{7.77}$$

3) Transverse equation: $$\lambda(t_f)=\frac{\partial\theta\left[x(t_f)\right]}{\partial x(t_f)} \tag{7.78}$$

We combine above positive equation and control equation to gain the optimal control law $u^*(t)$, optimal state trajectory curve $x^*(t)$ and optimal co-state trajectory curve $\lambda^*(t)$.

Example7.7 A state equation is as follows

$$\dot{x}(t)=u(t)$$

Initial condition is

$$x(t_0)=x_0$$

Please find optimal control law $u^*(t)$ which make the following performance index is minimum.

$$J=\frac{1}{2}cx^2(t_f)+\frac{1}{2}\int_{t_0}^{t_f}u^2dt,\quad c>0$$

Solution:

This is optimal problem of that final time t_f is given and that final state is free. For the control variable is not constrained, variation method may be employed to solve optimal problem. Constructing Hamilton function is

$$H(x,\quad u,\quad \lambda,\quad t)=\frac{1}{2}u^2+\lambda u$$

We can get from equation(7.76):

$$\dot{\lambda}=-\frac{\partial H}{\partial x}=-\frac{\partial}{\partial x}\left(\frac{1}{2}u^2+\lambda u\right)=0$$

Hence, $\lambda = Const$. In terms of transverse condition, we may gain:

$$\lambda(t_f)=\frac{\partial\theta[x(t_f)]}{\partial x(t_f)}=\frac{\partial}{\partial x(t_f)}\left[\frac{1}{2}cx^2(t_f)\right]=cx(t_f)$$

From control equation, we achieve

$$\frac{\partial H}{\partial u}=u+\lambda=0$$

Namely,

$$u^*=-\lambda=-cx(t_f)$$

Expression u^* is replaced into state equation, and we achieve

$$\dot{x}=u=-cx(t_f)$$

The solution of above differential equation is the following:

$$x(t)=-cx(t_f)(t-t_0)+x(t_0)$$

When $t=t_f$, the following expression may be acquired:

$$x(t_f)=-cx(t_f)(t_f-t_0)+x(t_0)$$

So that,

$$x(t_f)=\frac{x(t_0)}{1+c(t_f-t_0)}$$

Optimal control law is

$$u^*=-cx(t_f)=-\frac{cx(t_0)}{1+c(t_f-t_0)}$$

Case2, final time t_f is given and final state $x(t_f)$ is constrained.

We postulate that the final constraint is

$$M[x(t_f),\quad t_f]=M[x(t_f)]=0 \tag{7.79}$$

Final state $x(t_f)$ moves along the given boundary curve. Constructing augmented functional J_a is

$$\begin{aligned}J_a&=\theta[x(t_f)]+\upsilon^T M[x(t_f)]+\int_{t_0}^{t_f}\left\{L(x,\quad u,\quad t)+\lambda^T\left[f(x,\quad u,\quad t)-\dot{x}\right]\right\}dt\\&=\theta[x(t_f)]+\upsilon^T M[x(t_f)]+\int_{t_0}^{t_f}\left[H(x,\quad u,\quad t,\quad \lambda)-\lambda^T\dot{x}\right]dt\end{aligned} \tag{7.80}$$

Here, M, $\upsilon \in R$.

For augmented functional J_a, one-order variation is taken and ordered to be zero. The deducing course is similar to that of case1. The following expression could be acquired finally:

$$\delta J_a = \left[\frac{\partial\theta}{\partial x} + \left(\frac{\partial M}{\partial x}\right)^T \upsilon - \lambda\right]^T \delta x\Bigg|_{t=t_f} + \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}\right)^T \delta x + \left(\frac{\partial H}{\partial u}\right)^T \delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)^T \delta\lambda\right]dt = 0$$

(7.81)

Since variations $\delta x(t_f)$, δx, δu and $\delta\lambda$ are random and interdependent, one-order variation of augmented performance index functional J_a is zero, the necessary condition that equation(7.57) can gain extreme is

1) Positive equation

State equation: $\dot{x} = \dfrac{\partial H(x, \ u, \ \lambda, \ t)}{\partial \lambda}$ (7.81)

Adjoining equation: $\dot{\lambda} = -\dfrac{\partial H(x, \ u, \ \lambda, \ t)}{\partial x}$ (7.82)

2) Control equation: $\dfrac{\partial H(x, \ u, \ \lambda, \ t)}{\partial u} = 0$ (7.83)

3) Boundary condition: $x(t_0) = x_0$, $M[x(t_f)] = 0$ (7.84)

4) Transverse Condition: $\lambda(t_f) = \left[\dfrac{\partial\theta(x)}{\partial x} + \left(\dfrac{\partial M(x)}{\partial x}\right)^T \upsilon\right]\Bigg|_{t=t_f}$ (7.85)

Example7.8 A given state equation is as follows

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_1(t) + u(t)$$

Initial condition is

$$x_1(0) = 0, \qquad x_2(0) = 0$$

Final constraint condition is

$$x_1(3) + 4x_2(3) = 16$$

Please find optimal control law $u^*(t)$ which make the following performance index is minimum.

$$J = \frac{1}{2}[x_1(3) - 4]^2 + \frac{1}{2}[x_2(3) - 2]^2 + \frac{1}{2}\int_0^3 u^2(t)dt$$

Solution:

This problem belongs to the optimal control problem of case2. Here, variation method could be utilized to solve optimal solution of given problem for control variable $u(t)$ is not constrained. Similarly, constructing Hamilton function is

$$H = \frac{1}{2}u^2(t) + \lambda_1 x_2 - \lambda_2 x_1 + \lambda_2 u$$

$$\theta=\frac{1}{2}[x_1(3)-4]^2+\frac{1}{2}[x_2(3)-2]^2$$

$$M=x_1(3)+4x_2(3)-16=0$$

Confirm relative equations in the following:

State equation: $\dot{x}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\frac{\partial H}{\partial \lambda}=\begin{bmatrix}x_2\\ -x_1+u\end{bmatrix}$

Adjoining equation: $\dot{\lambda}=\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{bmatrix}=-\frac{\partial H}{\partial x}=\begin{bmatrix}\lambda_2\\ -\lambda_1\end{bmatrix}$

Control equation: $\frac{\partial H}{\partial u}=u+\lambda_2=0$

Hence, solve above equations and we can gain their solutions:

$\lambda_1=C_1\cos t+C_2\sin t$ $\qquad\lambda_2=-C_1\sin t+C_2\cos t$

$u=C_1\sin t-C_2\cos t$, $\qquad x_1(t)=C_4\cos t+C_5\sin t-\frac{C_2}{2}t\sin t-\frac{C_1}{2}t\cos t$

$$x_2(t)=-\left(\frac{C_2}{2}+C_4-\frac{C_1}{2}t\right)\sin t+\left(C_5-\frac{C_1}{2}-\frac{C_2}{2}t\right)\cos t$$

From initial condition $x_1(0)=0$ and $x_2(0)=0$, we can acquire:

$$C_4=0,\quad C_5-\frac{C_1}{2}=0$$

From final condition, the following expressions are gained:

$$x_1(3)=C_4\cos 3+C_5\sin 3-\frac{3}{2}C_2\sin 3-\frac{3}{2}C_1\cos 3$$

$$x_2(3)=-\left(\frac{1}{2}C_2+C_4-\frac{3}{2}C_1\right)\sin 3+\left(C_5-\frac{C_1}{2}-\frac{3}{2}C_2\right)\cos 3$$

In terms of transverse condition, we can know

$$\lambda_1(t_f)=\left[\frac{\partial\theta}{\partial x_1(t)}+\frac{\partial M}{\partial x_1(t)}\upsilon(t)\right]\Bigg|_{t=t_f=3}$$

namely, $\lambda_1(3)=x_1(3)-4+\upsilon=C_1\cos 3+C_2\sin 3$

$$\lambda_2(t_f)=\left[\frac{\partial\theta}{\partial x_2(t)}+\frac{\partial M}{\partial x_2(t)}\upsilon(t)\right]\Bigg|_{t=t_f=3}$$

namely, $\lambda_2(3)=x_2(3)-2+4\upsilon=-C_1\sin 3+C_2\cos 3$

Polynomials $x_1(3)$ and $x_2(3)$ are replaced into above equations, and then we can get

$$-\frac{5\cos 3}{2}C_1-\frac{5\sin 3}{2}C_2+C_4\cos 3+C_5\sin 3+\upsilon=4$$

$$\left(\frac{5}{2}\sin 3+\frac{1}{2}\cos 3\right)C_1-\left(\frac{1}{2}+\frac{5}{2}\cos 3\right)C_2-C_4\sin 3+C_5\cos 3+4\upsilon=2$$

Constraint equation from final constraint condition $M = x_1(3) + 4x_2(3) - 16 = 0$ is

$$(6\sin 3 - 3.5\cos 3)C_1 - (6\cos 3 + 3.5\sin 3)C_2 + (\cos 3 - 4\sin 3)C_4 + (4\cos 3 + \sin 3)C_5 = 16$$

Hence, the following equation group could be formed which unknown variables are respectively C_1, C_2, C_4, C_5 and υ.

$$C_4 = 0, \quad C_5 - \frac{C_1}{2} = 0$$

$$-\frac{5\cos 3}{2}C_1 - \frac{5\sin 3}{2}C_2 + C_4\cos 3 + C_5\sin 3 + \upsilon = 4$$

$$\left(\frac{5}{2}\sin 3 + \frac{1}{2}\cos 3\right)C_1 - \left(\frac{1}{2} + \frac{5}{2}\cos 3\right)C_2 - C_4\sin 3 + C_5\cos 3 + 4\upsilon = 2$$

$$(6\sin 3 - 3.5\cos 3)C_1 - (6\cos 3 + 3.5\sin 3)C_2 + (\cos 3 - 4\sin 3)C_4 + (4\cos 3 + \sin 3)C_5 = 16$$

Combine above equation group, and we can get the above solutions:

$$C_1 = 1.945, \quad C_2 = 2.08, \quad C_4 = 0, \quad C_5 = 0.9725, \quad \upsilon = -0.2172$$

Hence, optimal control law acquired is

$$u^*(t) = 1.945\sin t - 2.08\cos t$$

Case3, final time t_f is free and final state $x(t_f)$ is constrained.

The augmented functional J_a is

$$\begin{aligned} J_a &= \theta[x(t_f)] + \upsilon^T M[x(t_f)] + \int_{t_0}^{t_f}\left\{L(x, u, t) + \lambda^T\left[f(x, u, t) - \dot{x}\right]\right\}dt \\ &= \theta[x(t_f)] + \upsilon^T M[x(t_f)] + \int_{t_0}^{t_f}\left[H(x, u, t, \lambda) - \lambda^T\dot{x}\right]dt \end{aligned} \tag{7.86}$$

In equation(7.86), besides that time variable t_f is free, others are completely the same as that in equation(7.80). The necessary condition of optimal control problem is as follows:

1) Positive equation

State equation: $$\dot{x} = \frac{\partial H(x, u, \lambda, t)}{\partial \lambda} \tag{7.87}$$

Adjoining equation: $$\dot{\lambda} = -\frac{\partial H(x, u, \lambda, t)}{\partial x} \tag{7.88}$$

2) Control equation: $$\frac{\partial H(x, u, \lambda, t)}{\partial u} = 0 \tag{7.89}$$

3) Boundary condition: $$x(t_0) = x_0, \quad M[x(t_f), t_f] = 0 \tag{7.90}$$

4) Transverse Condition:

$$\lambda(t_f) = \left[\frac{\partial \theta(x)}{\partial x} + \left(\frac{\partial M(x)}{\partial x}\right)^T \upsilon\right]\Bigg|_{t=t_f} \tag{7.91}$$

$$\left[H(x, u, \lambda, t) + \frac{\partial \theta}{\partial t} + \upsilon^T\frac{\partial M}{\partial t}\right]_{t=t_f} = 0 \tag{7.92}$$

Example7.9 A given state equation is as follows

$$\dot{x}(t) = u(t)$$

Initial condition is $x(0) = x_0$; Final constraint condition is $x(t_f) = C_0(Const)$.Please find optimal control law $u^*(t)$ which make the following performance index is minimum.

$$J = \int_0^{t_f} \left(x^2 + \dot{x}^2 \right) dt$$

Solution:

The method of solving optimal control problem is similar to previous given method. Here, we also employed variation method to solve this example. Constructing Hamilton function is the following:

$$H(x,\quad u,\quad \lambda,\quad t) = x^2 + \dot{x}^2 + \lambda u = x^2 + u^2 + \lambda u$$

$$\theta[x(t)] = 0, \qquad M\left[x(t_f),\quad t_f\right] = 0$$

State equation:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = u$$

Adjoining equation:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x$$

Control equation:

$$\frac{\partial H}{\partial u} = 2u + \lambda = 0$$

Combine control equation and adjoining equation to gain the following equation:

$$x = \dot{u}$$

Above equation is replaced into state equation, and then we can know:

$$\ddot{u} = u$$

The solution of above second-order differential equation is

$$u(t) = C_1 e^t + C_2 e^{-t}$$

Then,

$$x(t) = C_1 e^t - C_2 e^{-t}, \qquad \lambda = -2C_1 e^t - 2C_2 e^{-t}$$

In terms of original condition and final constraint condition, we can get

$$C_1 - C_2 = x_0, \qquad C_1 e^{t_f} - C_2 e^{-t_f} = C_0$$

The solution of above equation group is

$$C_1 = \frac{C_0 - x_0 e^{-t_f}}{e^{t_f} - e^{-t_f}}, \qquad C_2 = \frac{C_0 - x_0 e^{t_f}}{e^{t_f} - e^{-t_f}}$$

Hence, optimal control law $u^*(t)$ and optimal state equation $x^*(t)$ are as follows:

$$u^*(t)=\frac{C_0-x_0e^{-t_f}}{e^{t_f}-e^{-t_f}}e^{t}+\frac{C_0-x_0e^{t_f}}{e^{t_f}-e^{-t_f}}e^{-t}, \quad x^*(t)=\frac{C_0-x_0e^{-t_f}}{e^{t_f}-e^{-t_f}}e^{t}-\frac{C_0-x_0e^{t_f}}{e^{t_f}-e^{-t_f}}e^{-t}$$

7.5 Pontryagin's Minimum Principle

In no constraint control, control law $u(t)$ is usually limited by any condition. However, in practice, control law $u(t)$ is always constrained by a certain objective condition. The control problem which control law $u(t)$ is limited is usually called to optimal control problem with constraint. Minimum principle will be introduced and employed to solve such optimal control problem in this section.

Minimum principle was put forward by Soviet expert Pontryagin in 1956. It was deduced from variation method; hence its conclusions are similar to the results of variation.However, there needs not to be Hamilton function for getting minimum, minimum principle may be widely applied into practice.

7.5.1 Minimum principle of continuous system

We postulate that state equation of system is as follows

$$\dot{x}=f[x(t),\quad u(t),\quad t],\quad x(t_0)=x_0 \tag{7.93}$$

In equation(7.93), $x\in R^n$, $u\in\Omega\in R^p$, Ω is closed set of boundary. Inequality constraint is

$$G[x(t),\quad u(t),\quad t]\geq 0 \tag{7.94}$$

In equation(7.94), G is m -dimension continuously differential vector function, $m\leq p$. System from original state x_0 to final state $x(t_f)$ requires that final state needs to satisfy the equality constraint:

$$M[x(t_f),\quad t_f]=0 \tag{7.95}$$

Here, , M is q -dimension continuously differential vector function, $q\leq n$. Performance index function is as follows:

$$J=\theta[x(t_f),\quad t_f]+\int_{t_0}^{t_f}L[x(t),\quad u(t),\quad t]dt \tag{7.96}$$

Optimal control is to find the optimally permissible control law $u(t)$ to make objective functional J have minimum.

In order to convert above inequality constraint to equality constraint, the following two variables are introduced:

1) Introduce a r -dimension control variable $\omega(t)$, and order,

$$\dot{\omega}(t)=u(t),\qquad \omega(t_0)=0 \tag{7.97}$$

Though $u(t)$ does not continue, $\omega(t)$ is continuous. If $u(t)$ is segmented and continuous, $\omega(t)$ is a smoothly continuous function.

2) Introduce another l-dimension variable $z(t)$, and order

$$\left[\dot{z}(t)\right]^2 = G[x(t),\ u(t),\ t], \qquad z(t_0)=0 \tag{7.98}$$

In spite of that $\dot{z}(t)$ is positive or negative, $\left[\dot{z}(t)\right]^2$ is eternally non-negative. Hence, it could satisfy the non-negative requirement. Through above transformation, inequality optimal problem could be converted into equality optimal problem. Then we introduces multipliers λ and γ by Lagrange multiplier method, above optimal problem will be further converted into confirm the extreme problem of augmented performance functional J_a.

$$\begin{aligned} J_a &= \theta[x(t_f),\ t_f] + \upsilon^T M[x(t_f),\ t_f] \\ &+ \int_{t_0}^{t_f}\left\{L\left(x,\ \dot{\omega},\ t\right) + \lambda^T\left[f\left(x,\ \dot{\omega},\ t\right) - \dot{x}\right] + \Gamma^T\left[G\left(x,\ \dot{\omega},\ t\right) - \left(\dot{z}\right)^2\right]\right\}dt \\ &= \theta[x(t_f),\ t_f] + \upsilon^T M[x(t_f),\ t_f] \\ &+ \int_{t_0}^{t_f}\left\{H\left[x,\ \dot{\omega},\ \lambda,\ t\right] - \lambda^T\dot{x} + \Gamma^T\left[G\left(x,\ \dot{\omega},\ t\right) - \left(\dot{z}\right)^2\right]\right\}dt \end{aligned} \tag{7.99}$$

Here, Hamilton function $H\left(x,\ \dot{\omega},\ \lambda,\ t\right)$ is defined as follows:

$$H\left(x,\ \dot{\omega},\ \lambda,\ t\right) = L\left(x,\ \dot{\omega},\ t\right) + \lambda^T f\left(x,\ \dot{\omega},\ t\right) \tag{7.100}$$

In order to simplify above problem, Lagrange scalar function ϕ is defined as follows:

$$\phi\left(x,\ \dot{x},\ \dot{\omega},\ \dot{z},\ \lambda,\ \Gamma,\ t\right) = H\left(x,\ \dot{\omega},\ \lambda,\ t\right) - \lambda^T\dot{x} + \Gamma^T\left[G\left(x,\ \dot{\omega},\ t\right) - \left(\dot{z}\right)^2\right] \tag{7.101}$$

Then, augmented performance functional J_a could be written into

$$J_a = \theta[x(t_f),\ t_f] + \upsilon^T M[x(t_f),\ t_f] + \int_{t_0}^{t_f}\Phi\left(x,\ \dot{x},\ \dot{\omega},\ \dot{z},\ \lambda,\ \Gamma,\ t\right)dt \tag{7.102}$$

One-order variation of augmented performance functional J_a is exerted and then gained:

$$\delta J_a = \left[\Phi + \frac{\partial\theta}{\partial t} + \upsilon^T \frac{\partial M}{\partial t}\right]_{t=t_f^*} \delta t_f + \left[\left(\frac{\partial\theta}{\partial x}\right)^T + \upsilon^T \frac{\partial M}{\partial x}\right]_{t=t_f^*} \delta x(t_f^*)$$
$$+\int_{t_0}^{t_f^*}\left[\left(\frac{\partial\Phi}{\partial x}\right)^T \delta x + \left(\frac{\partial\Phi}{\partial \dot{x}}\right)^T \delta \dot{x} + \left(\frac{\partial\Phi}{\partial \dot{\omega}}\right)^T \delta \dot{\omega} + \left(\frac{\partial\Phi}{\partial \dot{z}}\right)^T \delta \dot{z}\right] dt \tag{7.103}$$

Here, t_f^* is optimally final time. The backward three terms in above integrated term are partially integrated respectively, and the following relationship is utilized:

$$\delta x(t_f) = \delta x(t_f^*) + \dot{x}(t_f)\delta t_f \tag{7.104}$$

Then we can acquire

$$\delta J_a = \left[\Phi - \left(\dot{x}\right)^T \left(\frac{\partial\Phi}{\partial \dot{x}}\right) + \frac{\partial\theta}{\partial t} + \upsilon^T \frac{\partial M}{\partial t}\right]_{t=t_f^*} \delta t_f + \left[\frac{\partial\theta}{\partial x} + \left(\frac{\partial M}{\partial x}\right)^T \upsilon + \frac{\partial\Phi}{\partial \dot{x}}\right]_{t=t_f^*}^T \delta x(t_f^*)$$
$$+\left[\left(\frac{\partial\Phi}{\partial \dot{\omega}}\right)^T \delta\omega + \left(\frac{\partial\Phi}{\partial \dot{z}}\right)^T \delta z\right]_{t=t_f^*} + \int_{t_0}^{t_f^*}\left[\left(\frac{\partial\Phi}{\partial x} - \frac{d}{dt}\frac{\partial\Phi}{\partial \dot{x}}\right)^T \delta\omega - \left(\frac{d}{dt}\frac{\partial\Phi}{\partial \dot{\omega}}\right)^T \delta\omega - \left(\frac{d}{dt}\frac{\partial\Phi}{\partial \dot{z}}\right)^T \delta z\right] dt$$
$$\tag{7.105}$$

According to the necessary condition of functional extreme, the following should be found:

$$\delta J_a = 0 \tag{7.106}$$

In equation(7.105), variations δt_f, $\delta x(t_f^*)$, δx, $\delta\omega$, δz are both random and independent, hence the necessary condition that augmented performance functional J_a gets extreme is the following:

$$\left[\Phi - \left(\dot{x}\right)^T \left(\frac{\partial\Phi}{\partial \dot{x}}\right) + \frac{\partial\theta}{\partial t} + \upsilon^T \frac{\partial M}{\partial t}\right]_{t=t_f^*} = 0 \tag{7.107a}$$

$$\left[\frac{\partial\theta}{\partial x} + \left(\frac{\partial M}{\partial t}\right)^T \upsilon + \frac{\partial\Phi}{\partial \dot{x}}\right]_{t=t_f^*} = 0 \tag{7.107b}$$

$$\left(\frac{\partial\Phi}{\partial \dot{\omega}}\right)_{t=t_f^*} = 0, \quad \left(\frac{\partial\Phi}{\partial \dot{z}}\right)_{t=t_f^*} = 0 \tag{7.107c}$$

$$\frac{\partial\Phi}{\partial x} - \frac{d}{dt}\frac{\partial\Phi}{\partial \dot{x}} = 0 \tag{7.107d}$$

$$\frac{d}{dt}\frac{\partial\Phi}{\partial \dot{\omega}} = 0, \quad \frac{d}{dt}\frac{\partial\Phi}{\partial \dot{z}} = 0 \tag{7.107e}$$

From equation(7.101), gain

$$\frac{\partial \Phi}{\partial x}=\frac{\partial H}{\partial x}+\left(\frac{\partial G}{\partial x}\right)^T \Gamma, \qquad \frac{\partial \Phi}{\partial \dot{x}}=-\lambda$$

Above is substituted into equation(7.107d) and then we can acquire:

$$-\frac{d\lambda}{dt}=\frac{\partial H}{\partial x}+\left(\frac{\partial G}{\partial x}\right)^T \Gamma \tag{7.108a}$$

Namely,

$$\dot{\lambda}=\frac{d\lambda}{dt}=-\frac{\partial H}{\partial x}-\left(\frac{\partial G}{\partial x}\right)^T \Gamma \tag{7.108b}$$

If there is no state variable x in inequality constraint function G,

$$G[u(t),\quad t]\geq 0$$

we can get the following expression from equation(7.108b) for $\frac{\partial G}{\partial x}\equiv 0$

$$\dot{\lambda}=-\frac{\partial H}{\partial x} \tag{7.109}$$

From equation(7.107a) and equation(7.107b) and $\frac{\partial \Phi}{\partial \dot{x}}=-\lambda$, we can calculate the value λ and H when $t=t_f^*$:

$$\lambda(t_f^*)=\left[\frac{\partial \theta}{\partial x}+\left(\frac{\partial M}{\partial t}\right)^T \upsilon\right]_{t=t_f^*} \tag{7.110}$$

$$H(t_f^*)=\left[-\frac{\partial \theta}{\partial t}-\upsilon^T\frac{\partial M}{\partial t}\right]_{t=t_f^*} \tag{7.111}$$

When system state under optimal control $u^*(t)$ is transformed along optimal trajectory $x^*(t)$, the asterisk(*) of optimal time t_f^* can be neglected from equation(7.110) and equation(7.111). And then we can get transverse condition:

$$\lambda(t_f)=\left[\frac{\partial \theta}{\partial x}+\left(\frac{\partial M}{\partial t}\right)^T \upsilon\right]_{t=t_f} \tag{7.112}$$

$$\left[H(x,\quad u,\quad \lambda,\quad t)+\frac{\partial \theta}{\partial t}+\upsilon^T\frac{\partial M}{\partial t}\right]_{t=t_f}=0 \tag{7.113}$$

Expression(7.107e) demonstrates that derivatives $\frac{\partial \Phi}{\partial \dot{\omega}}$ and $\frac{\partial \Phi}{\partial \dot{z}}$ are constant along optimal trajectory line. From equation(7.107c), it is known that these constants are zero; hence optimal trajectory curve is

$$\frac{\partial \Phi}{\partial \dot{\omega}}=\frac{\partial \Phi}{\partial \dot{z}}\equiv 0 \tag{7.114}$$

Since Lagrange function Φ includes variables $\dot{x}$, $\dot{\omega}$ and $\dot{z}$, if variables $\dot{x}$, $\dot{\omega}$ and $\dot{z}$ in the extreme curve could be respectively expressed by optimal variables $\dot{x}^*$, $\dot{\omega}^*$ and $\dot{z}^*$, equation(7.114) can be expressed as follows:

$$\frac{\partial \Phi}{\partial \dot{\omega}^*} = \frac{\partial \Phi}{\partial \dot{z}^*} \equiv 0 \tag{7.115}$$

Above content is the necessary condition that performance index functional J_a gets extreme. Besides above necessary conditions, we need to confirm its sufficient condition: Weierstrass function is non-negative function along optimal trajectory curve, namely:

$$\begin{aligned} E &= \Phi\left(x^*,\ \dot{x},\ \dot{\omega},\ \dot{z},\ \lambda^*,\ \Gamma^*,\ t\right) - \Phi\left(x^*,\ \dot{x}^*,\ \dot{\omega}^*,\ \dot{z}^*,\ \lambda^*,\ \Gamma^*,\ t\right) \\ &- \left(\frac{\partial \Phi}{\partial \dot{x}^*}\right)^T \left(\dot{x} - \dot{x}^*\right) \\ &= \Phi\left(x^*,\ \dot{x},\ \dot{\omega},\ \dot{z},\ \lambda^*,\ \Gamma^*,\ t\right) \\ &+ \lambda^{*T}\dot{x} - \Phi\left(x^*,\ \dot{x}^*,\ \dot{\omega}^*,\ \dot{z}^*,\ \lambda^*,\ \Gamma^*,\ t\right) - \lambda^{*T}\dot{x}^* \\ &= H\left(x^*,\ \dot{\omega},\ \lambda^*,\ t\right) - H\left(x^*,\ \dot{\omega}^*,\ \lambda^*,\ t\right) \geq 0 \end{aligned} \tag{7.116}$$

Expressions $\dot{\omega}(t) = u(t)$, $\dot{\omega}^*(t) = u^*(t)$ is replaced into above expression and then gain:

$$H\left(x^*,\ u^*,\ \lambda^*,\ t\right) \leq H\left(x^*,\ u,\ \lambda^*,\ t\right) \tag{7.117}$$

Lagrange function expression Φ is substituted into $\frac{\partial \Phi}{\partial \dot{\omega}} = 0$, and gain:

$$\frac{\partial H}{\partial \dot{\omega}} + \left(\frac{\partial G}{\partial \dot{\omega}}\right)^T \Gamma = 0 \tag{7.118}$$

If the expression $\dot{\omega}(t) = u(t)$ is noted, the following expression can be gained:

$$\frac{\partial H}{\partial u} + \left(\frac{\partial G}{\partial u}\right)^T \Gamma = 0 \tag{7.119}$$

After we summarize above deduction, the following famous minimum principle can be achieved.

Principle of the Minimum Assume the state equation of system is as follows:

$$\dot{x} = f[x(t),\ u(t),\ t], \qquad x(t_0) = x_0 \tag{7.120}$$

Control $u(t)$ is a segmented and continuous function with the first discontinuity point, which belongs to the p-dimension boundary set Ω and furthermore satisfy the following inequality:

$$G[x(t),\quad u(t),\quad t] \geq 0 \tag{7.121}$$

In the case of unknown final time t_f, system state $x(t)$ satisfy the boundary condition:

$$M[x(t_f),\quad t_f] = 0 \tag{7.122}$$

Furthermore system state could be transformed from original state $x(t_0)$ to final state $x(t_f)$ and make the following performance index functional have minimum.

$$J = \theta[x(t_f),\quad t_f] + \int_{t_0}^{t_f} L[x(t_f),\quad u(t),\quad t]dt \tag{7.123}$$

Then optimal control $u^*(t)$, optimal trajectory $x^*(t)$ and optimal adjoining vector $\lambda^*(t)$ have to satisfy the following conditions:

Assume that Hamilton function is the following:

$$H(x,\quad u,\quad \lambda,\quad t) = L(x,\quad u,\quad t) + \lambda^T f(x,\quad u,\quad t) \tag{7.124}$$

1) Satisfy canonical equation along optimal trajectory curve $x^*(t)$

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{7.125}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} - \left(\frac{\partial G}{\partial x}\right)^T \Gamma \tag{7.126}$$

Here, sign Γ is Lagrange multiplier vector which is not related to time t and which dimension is the same as that of inequality constraint function G. If inequality constraint function G does not include state variable x, then we have

$$G[u(t),\quad t] \geq 0 \tag{7.127}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{7.128}$$

2) Transverse condition and boundary condition

$$\lambda(t_f) = \left[\frac{\partial \theta}{\partial x} + \left(\frac{\partial M}{\partial x}\right)^T \upsilon\right]_{t=t_f} \tag{7.129}$$

$$\left[H(x,\quad u,\quad \lambda,\quad t) + \frac{\partial \theta}{\partial t} + \left(\frac{\partial M}{\partial t}\right)^T \upsilon\right]_{t=t_f} = 0 \tag{7.130}$$

$$x(t_0) = x_0 \qquad M[x(t_f),\quad t_f] = 0 \tag{7.131}$$

3) In optimal trajectory $x^*(t)$, the Lagrange function H corresponding to optimal control $u^*(t)$ can get absolute minimum, namely:

$$H(x^*,\quad u^*,\quad \lambda^*,\quad t) = \min_{u\in\Omega} H(x^*,\quad u,\quad \lambda^*,\quad t) \tag{7.132}$$

Or written into

$$H\left(x^*,\ u^*,\ \lambda^*,\ t\right) \le \min_{u\in\Omega} H\left(x^*,\ u,\ \lambda^*,\ t\right) \qquad (7.133)$$

Furthermore, the following equation is always found along optimal trajectory curve $x^*(t)$:

$$\frac{\partial H}{\partial u} = -\left(\frac{\partial G}{\partial u}\right)^T \Gamma \qquad (7.134)$$

Above minimum principle is also called to maximum principle for some engineering problem requires that performance indexes is maximum. Then, so long as the sign L in Hamilton function H is changed into $-L$ and adjoining vector λ changes its sign and furthermore Hamilton function H is maximum, minimum principle can become maximum principle.

Example7.10 A given state equation is as follows

$$\dot{x} = x - u\,,\quad x(0) = 5$$

Control constraint condition: $\frac{1}{2} \le u \le 1$.

Please find optimal control law $u^*(t)$ which makes the following performance index is minimum.

$$\min J = \int_0^1 (x+u)dt$$

Solution: This is a permissible control problem which final state is free.

1) Construct Hamilton function

From equation(7.124), the Hamilton function H could be written out:

$$H = L + \lambda^T f = (x+u) + \lambda(x-u)$$

Obviously, the Hamilton function is linear function of control variable u. Derivative $\frac{\partial H}{\partial u} = 1-\lambda$ is not related to control variable u. In terms of minimum principle, finding minimum of Hamilton function H is equivalent to finding the minimum of functional J; this requires that the expression $\lambda(1-u)$ is minimum.

Given constraint of control variable u: $\frac{1}{2} \le u \le 1$.

Hence, when $\lambda > 1$, the upper bound of control variable u: $u^* = 1$;

when $\lambda < 1$, the lower bound of control variable u: $u^* = \frac{1}{2}$.

2) Find adjoining vector $\lambda(t)$ to confirm the converting point.

From co-state equation(7.128), we can know:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1 - \lambda$$

Namely, $\quad \dot{\lambda} + \lambda = -1$

The solution of above differential equation is: $\lambda(t) = -1 + Ce^{-t}$.

When $t_f = 1$, $\lambda(t_f) = \lambda(1) = 0$, $C = e$; hence, $\lambda(t) = e^{1-t} - 1$.

Ensure the converting point:

We order $\lambda = 1$, and then can get: $t = 1 - In2 \approx 0.307$.

When $\lambda > 1$ ($t < 0.307$), $u^* = 1$.

When $\lambda < 1$ ($t > 0.307$), $u^* = \frac{1}{2}$.

3) Acquire optimal state trajectory curve $x^*(t)$

Given system state equation: $\dot{x} = x - u$

The solution of above state equation is: $x = u + C_1 e^t$.

When $0 \le t < 0.307$, $u^* = 1$; hence $x = 1 + C_1 e^t$.On consider $x(0) = 5$, we can get the optimal state trajectory curve: $x^*(t) = 1 + 4e^t$.

When $0.307 < t \le 1$, $u^* = \frac{1}{2}$; hence $x = \frac{1}{2} + C_2 e^t$. For first segment value of system state $x(0.307) = 6.438$ is the original value of second segment of state curve, we can achieve the optimal state trajectory curve: $x^*(t) = 4.368e^t + 0.5$.

4) Confirm minimum functional $J^* = J(u^*)$.

$$J^* = \int_0^1 (x+u)dt = \int_0^{0.307} (x+1)dt + \int_{0.307}^1 (x+0.5)dt$$
$$= \int_0^{0.307} (4e^t + 2)dt + \int_{0.307}^1 (4.368e^t + 1)dt = 8.684$$

7.5.2 Minimum principle of discrete system

The method of finding the optimal control in discrete system is similar to that in continuous system, which corresponding relationship is provided in table7.2.

Example7.11 A given state equation of discrete system is as follows

$$x(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(k)$$

Control boundary condition is as follows:

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x(2) = \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Please find optimal control sequence $u^*(k)$ and optimal state sequence $x^*(k)$ which make the following performance index is minimum.

$$J = 0.05 \sum_{k=0}^{1} u^2(k)$$

Solution:

1) Construct Hamilton function H

Table7.2 Comparing table of Minimum principle between discrete and continuous systems

	Minimum principle of continuous system	Minimum principle of discrete system
System	$\dot{x} = f[x(t),\ u(t),\ t]$, $x(t_0) = x_0$	$x(k+1) = f[x(k),\ u(k),\ k]$ $x(0) = x_0$, $k = 0,1,\cdots\cdots,N$
Performance index	$J = \theta[x(t_f),\ t_f] + \int_{t_0}^{t_f} L[x(t),\ u(t),\ t]dt$	$J = \theta[x(N),\ N] + \sum_{k=0}^{N-1} L[x(k),\ u(k),\ k]$
Extreme problem	Find optimal control $u^*(t)$ which makes functional J have minimum.	Find optimal control $u^*(k),\ k = 0,1,2\cdots N-1$ which makes functional J have minimum.
Method features	Cite adjoining vector $\lambda(t)$	Cite adjoining vector sequence $\lambda(k),\ k = 0,1,2,\cdots N$
Hamilton function	$H(x,\ u,\ \lambda,\ t) = L(x,\ u,\ t) + \lambda^T f(x,\ u,\ t)$	$H(k) = L[x(k),\ u(k),\ k] + \lambda^T(k+1) f[x(k),\ u(k),\ k]$ $k = 0,1,2\cdots N-1$
Canonical equation	$\dot{x} = \frac{\partial H}{\partial \lambda}$, $\dot{\lambda} = -\frac{\partial H}{\partial x}$	$x(k+1) = \frac{\partial H(k)}{\partial \lambda(k+1)}$ $\lambda(k) = \frac{\partial H(k)}{\partial x(k)}$ $k = 0,1,\cdots N-1$
Extreme condition(unconstraint control)	$\frac{\partial H}{\partial u} = 0$	$\frac{\partial H(k)}{\partial u(k)}$, $k = 0,1,2\cdots N-1$
Extreme condition(constraint control)	$H(x^*,\ u^*,\ \lambda^*,\ t) = \min_{\Omega} H(x^*,\ u,\ \lambda^*,\ t)$	$H[x^*(k),\ u^*(k),\ \lambda^*(k+1),\ k] = \min_{\Omega} H[x^*(k),\ u(k),\ \lambda^*(k+1),\ k]$ $k = 0,1,2,\cdots N-1$
Transverse condition (final state is free and final time is given)	$\lambda(t_f) = \frac{\partial \theta}{\partial x(t_f)}$ When $\theta[x(t_f),\ t_f] = 0$, $\lambda(t_f) = 0$	$\lambda(N) = \frac{\partial \theta}{\partial x(N)}$ When $\theta[x(N),\ N] = 0$, $\lambda(N) = 0$

$$H(k) = L[x(k),\ u(k),\ k] + \lambda^T(k+1) f[x(k),\ u(k),\ k]$$
$$= 0.05u^2(k) + \lambda^T(k+1)[Ax(k) + Bu(k)]$$

Adjoining equation is as follows:

$$\lambda(k) = \frac{\partial H(k)}{\partial x(k)} = A^T \lambda(k+1)$$

Control equation is

$$\frac{\partial H(k)}{\partial u(k)} = 0.1u(k) + B^T\lambda(k+1) = 0$$

Hence, control law expression $u(k)$ can be achieved:

$$u(k) = -10B^T A^{-T}\lambda(k) = [0.1 \quad -1]\lambda(k)$$

When $k=0$, $\lambda(0) = A^T\lambda(1)$

$$x(1) = Ax(0) + Bu(0) = Ax(0) - 10BB^T A^{-T}\lambda(0) = Ax(0) - 10BB^T\lambda(0)$$

$$= \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0.1\end{bmatrix}\lambda(1)$$

When $k=1$, $\lambda(1) = A^T\lambda(2)$

$$x(2) = Ax(1) + Bu(1) = Ax(1) - 10BB^T A^{-T}\lambda(1)$$

$$= A^2x(0) - 10ABB^T\lambda(1) - 10ABB^T A^{-T}\lambda(1)$$

$$= \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & 0.01\\0 & 0.1\end{bmatrix}\lambda(1) - \begin{bmatrix}0 & 0\\-0.01 & 0.1\end{bmatrix}\lambda(1)$$

$$= \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & 0.01\\-0.01 & 0.2\end{bmatrix}\lambda(1) = \begin{bmatrix}1-0.01\lambda_2(1)\\0.01\lambda_1(1)-0.2\lambda_2(1)\end{bmatrix}$$

Boundary condition is the following from given information in this example:

$$x(2) = \begin{bmatrix}x_1(2)\\x_2(2)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Then we can get

$$1-0.01\lambda_2(1) = 0, \qquad 0.01\lambda_1(1) - 0.2\lambda_2(1) = 0$$

The solution of above equation group is

$$\lambda_1(1) = 2000, \quad \lambda_2(1) = 100$$

Hence,

$$\lambda(0) = A^T\lambda(1) = \begin{bmatrix}1 & 0\\0.1 & 1\end{bmatrix}\begin{bmatrix}2000\\100\end{bmatrix} = \begin{bmatrix}2000\\300\end{bmatrix}$$

$$x(1) = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0.1\end{bmatrix}\lambda(1) = \begin{bmatrix}1\\-10\end{bmatrix}$$

$$u(0) = [0.1 \quad -1]\lambda(0) = -100$$

$$u(1) = [0.1 \quad -1]\lambda(1) = 100$$

Relative results are listed as follows:

$$x(0) = \begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad x(1) = \begin{bmatrix}x_1(1)\\x_2(1)\end{bmatrix} = \begin{bmatrix}0\\-10\end{bmatrix}, \quad x(2) = \begin{bmatrix}x_1(1)\\x_2(1)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\lambda(0) = \begin{bmatrix}2000\\300\end{bmatrix}, \quad \lambda(1) = \begin{bmatrix}2000\\100\end{bmatrix}, \quad u(0) = -100, \quad u(1) = 100$$

7.6 Bang-bang Control(Switching Control)

It's very convenient to utilize minimum principle to solve optimal control; however it's very difficult to give out specific control law $u^*(t)$. In the following paragraphs, a kind of special case will be discussed. In this case, all components of control vector can get the boundary value of control field; and furthermore they could be continuously switched from one boundary value to another boundary value; and then a strongest control will be formed and constructed. Such control is called to switch control. If such control is simulated and visualized, such control is also called to Bang-bang control. Time optimization is a classical example of Bang-bang control. For the performance indexes of such control is very simple and studied earlier, relative research results is very abundant and prolific.

For non-linear system, system state equation is as follows:

$$\begin{cases} \dot{x} = f[x(t), \quad t] + G[x(t), \quad t]u(t) \\ x(t_0) = x_0 \end{cases} \tag{7.135}$$

In equation(7.135), $a_i \le u_i \le b_i$. Find the optimal control law $u^*(t)$ to make the following performance index function have minimum.

$$J = \theta[x(t_f), \quad t_f] + \int_{t_0}^{t_f} \{\phi[x(t), \quad t] + h^T[x(t), \quad t]u(t)\}dt \tag{7.136}$$

Above optimal problem has the following features:

(1) given state equation is nonlinear for system state vector $x(t)$ in equation(7.135); but for control vector $u(t)$ shows linear relationship. Such system is called to nonlinear radiation system.

(2) the relationship between aim function and control variable $u(t)$ in equation (7.136). For the derivative of Hamilton function to input control variable $u(t)$ causes that there is no control variable $u(t)$ in the derivative equation, the relationship among control law $u(t)$, system state $x(t)$ and adjoining vector $\lambda(t)$ can not be gotten from coupling equation.

(3) Control law $u(t)$ is limited.

In order to solve such optimal control problem, now let's firstly define Hamilton function:

$$H[x(t), \quad u(t), \quad \lambda(t), \quad t] = \phi[x(t), \quad t] + h^T[x(t), \quad t]u(t) + \lambda^T(t)\{f[x(t), \quad t] + G[x(t), \quad t]u(t)\} \tag{7.136}$$

Since Hamilton function H is linear for control law $u(t)$, minimum requirement of Hamilton function to control law $u(t)$ is as follows:

$$u_i = \begin{cases} a_i & \{h^T[x(t), \quad t] + \lambda^T(t)G[x(t), \quad t]\}_i > 0 \\ b_i & \{h^T[x(t), \quad t] + \lambda^T(t)G[x(t), \quad t]\}_i < 0 \end{cases} \tag{7.137}$$

Obviously, control law $u(t)$ linearly appears in system and performance indexes. If each component of control vector is boundary, such control is Bang-bang control.

Only when the following expression is found

$$h^T[x(t),\quad t]+\lambda^T(t)G[x(t),\quad t]=0 \tag{7.138}$$

Hamilton function will not be the function of control law $u(t)$. And it will not have minimum for control law $u(t)$.

For Bang-bang control problem, there are the following state equation and adjoining equation:

State equation:

$$\dot{x}=\frac{\partial H}{\partial \lambda}=f[x(t),\quad t]+G[x(t),\quad t]u(t) \tag{7.139}$$

Adjoining equation:

$$-\dot{\lambda}(t)=\frac{\partial H}{\partial x}=\frac{\partial \phi[x(t),\quad t]}{\partial x}+\frac{\partial h^T[x(t),\quad t]}{\partial x}u(t)+\frac{\partial f^T[x(t),\quad t]}{\partial x}\lambda(t)+\frac{\partial \{G[x(t),\quad t]u(t)\}^T}{\partial x}\lambda(t) \tag{7.140}$$

In equation(7.140), control law $u(t)$ is decided by equation(7.137).

Above state equation, adjoining equation and boundary condition constitute two boundary value problem together; this is a problem solved difficultly. Here, we only consider a kind of special case namely shortest time problem.

Assume that the state equation of controllable linear time-constant control system is as follows:

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{7.140}$$

State constraint condition: $x(t_0)=x_0$, $x(t_f)=0$, final time t_f is not certain.

Performance index function:

$$\min_u J=\int_0^{t_f} dt \tag{7.141}$$

Control constraint condition:

$$|u_i(t)|\le 1 \quad i=1,2,\cdots,r \tag{7.142}$$

Search optimal control $u^*(t)$ to make that system state can be transformed in shortest period from original state $x(t_0)$ to original point $x(t_f)=0$.

Here, $\theta[x(t),\quad u(t),\quad t]=0$ and $L[x(t),\quad u(t),\quad t]=1$; hence Hamilton function is

$$H[x(t),\quad u(t),\quad \lambda,\quad t]=1+\lambda^T[Ax(t)+Bu(t)]=1+x^TA^T\lambda+u^TB^T\lambda \tag{7.143}$$

In order to make that Hamilton function H is globally minimum, it's known that optimal control is as follows:

$$u^*(t)=-SGN[B^T\lambda(t)],\ \text{or,}\ \ u_i^*(t)=-\text{sgn}[B^T\lambda(t)]_i,\quad i=1,2,\cdots,r \tag{7.144}$$

Here, sgn is sign function which is defined as

$$\operatorname{sgn}\alpha=\begin{cases}+1 & \alpha>0\\ 0 & \alpha=0\\ -1 & \alpha<0\end{cases} \tag{7.145}$$

When α is vector, sign function is expressed by SGN.

Canonical equation group:

$$\dot{\lambda}=-\frac{\partial H}{\partial x}=-A^T\lambda \tag{7.146a}$$

$$\dot{x}(t)=\frac{\partial H}{\partial \lambda}=Ax(t)+Bu(t) \tag{7.146b}$$

Solve and gain:

$$\lambda(t)=e^{-A^Tt}\lambda(0) \tag{7.147}$$

Here, $\lambda(0)=\lambda_0$, it is a non-zero vector. Otherwise, there will be a wrong result.

Equation(7.147) is replaced into equation(7.144), and then achieve the following equation:

$$u^*(t)=-SGN\left[B^Te^{-A^Tt}\lambda_0\right] \tag{7.148}$$

Apparently, optimal control of time is switch control (Bang-bang control). Optimal control of time requires that control variable only gets boundary value(maximum value) but have opposite sign with adjoining vector $\lambda(t)$.

Now let's discuss the uniquity of time optimal control and switch degree.

Theorem For linear time-constant system $\sum(A,\quad B,\quad C)$, if there is time optimal control $u^*(t)$, such control $u_i(t),\quad i=1,2,\cdots,r$ is unique.

Demonstration Here we employ proof of contradiction to prove the theorem.

Assume that there have two control vectors u_1 and u_2, $u_1\neq u_2$; but they can be transformed in the shortest time from original state $x(t_0)$ to original point $x(t_f)$.

In terms of minimum principle, u_1 and u_2 can both make that Hamilton H is globally minimum; they are that one is greater, another is lower, or they are equal. Now assume that u_1 can make H be less. From equation(7.143), we can gain:

$$1+\lambda_0^Te^{-At}\left[Ax(t)+Bu_1(t)\right]\leq 1+\lambda_0^Te^{-At}\left[Ax(t)+Bu_2(t)\right]$$

Namely,

$$\lambda_0^Te^{-At}Bu_1(t)\leq\lambda_0^Te^{-At}Bu_2(t) \tag{7.149}$$

On the other hand, in the viewpoint of state equation solution, there is

$$x_1(t)=e^{At}x(0)+\int_0^{t_f}e^{A(t-\tau)}Bu_1(\tau)d\tau$$

$$x_2(t)=e^{At}x(0)+\int_0^{t_f}e^{A(t-\tau)}Bu_2(\tau)d\tau$$

When $t=t_f$, system state vectors $x_1(t)$ and $x_2(t)$ are both converted into original point; namely, $x_1(t_f)=x_2(t_f)=0$. Then we have

$$\int_0^{t_f} e^{A(t_f-\tau)}Bu_1(\tau)d\tau=\int_0^{t_f} e^{A(t_f-\tau)}Bu_2(\tau)d\tau$$

For $\left(e^{At_f}\right)^{-1}=e^{-At_f}$, above equation can be simplified into

$$\int_0^{t_f} e^{-At}Bu_1(\tau)d\tau=\int_0^{t_f} e^{-A\tau}Bu_2(\tau)d\tau$$

After variable is substituted, we can gain:

$$\int_0^{t_f} e^{-At}Bu_1(t)dt=\int_0^{t_f} e^{-At}Bu_2(t)dt$$

Adjoining vector value λ_0^T is multiplied into the both sides of above equation.

$$\int_0^{t_f} \lambda_0^T e^{-At}Bu_1(t)dt=\int_0^{t_f} \lambda_0^T e^{-At}Bu_2(t)dt \tag{7.150}$$

Inequality(7.149) is the condition of ensuring that Hamilton function H is minimum, and equation(7.150) is the condition of requiring that final state is zero. For these two equations have to been satisfied, there is

$$\lambda_0^T e^{-At}Bu_1(t)=\lambda_0^T e^{-At}Bu_2(t) \tag{7.151}$$

In the case of that given system $\sum(A,\quad B,\quad C)$ is controllable, only possible case is as follows:

$$u_1(t)=u_2(t) \tag{7.152}$$

Above equation(7.152) demonstrates that control vector $u(t)$ is unique. Above theorem has been proved.

Theorem For linear time-constant system $\sum(A,\quad B,\quad C)$, if there is time optimal control $u^*(t)$ which satisfies $|u_i|\le 1,\quad i=1,2,\cdots,r$, and if all eigenvalues of system matrix A are both real, each control $u_i(t),\quad i=1,2,\cdots,r$ is Bang-bang control, and furthermore the most degree between two boundary values is $n-1$.

If all eigenvalues of linear time-constant system is non-positive real, control law $u(t)$ is permissible control; there must have optimal control of time.

Example7.12 A given state equation of linear time-constant system is as follows

$$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=u(t)\end{cases}$$

Control constraint condition is as follows:

$$|u(t)|\le 1$$

Please find optimal control $u^*(t)$ which makes system be transformed from original state $x(0)$ to original point.

Solution:

From the given information of this example, we can write out performance index functional J :

$$J=\int_0^{t_f} dt=t_f$$

Construct Hamilton function H :

$$H(x,\quad u,\quad \lambda,\quad t)=L+\lambda^T f=1+\lambda_1 x_2+\lambda_2 u$$

From minimum principle, we can achieve adjoining equation:

$$\dot{\lambda}=\begin{bmatrix}\dot{\lambda}_1(t)\\ \dot{\lambda}_2(t)\end{bmatrix}=-\frac{\partial H}{\partial x}=-\begin{bmatrix}0\\ \lambda_1(t)\end{bmatrix}$$

Then the solution of above differential equation group is as follows:

$$\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\end{bmatrix}=\begin{bmatrix}C_1\\ C_2-C_1 t\end{bmatrix}$$

The optimal control which make Hamilton function H have minimum is

$$u^*(t)=-\operatorname{sgn}[\lambda_2(t)]=-\operatorname{sgn}(C_2-C_1 t)$$

$$x(t_f)=0$$

For $|u(t)|\le 1$, above optimal control are divided into two cases for discussion:

1) when $u(t)=+1$,

$$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=1\end{cases}$$

The solution of above differential equation group is as follows:

$$x_2(t)=t+x_2(0),\quad x_1(t)=\frac{t^2}{2}+x_2(0)t+x_1(0)$$

Eliminate time variable t to gain:

$$x_1(t)=\frac{x_2^2(t)}{2}+x_1(0)-\frac{x_2^2(0)}{2}$$

The optimal trajectory in phase plane can be described as the following expression:

$$x_1(t)=\frac{x_2^2(t)}{2}+c$$

2) when $u(t)=-1$,

$$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=-1\end{cases}$$

The solution of above differential equation group is as follows:

$$x_2(t)=-t+x_2(0),\quad x_1(t)=-\frac{t^2}{2}+x_2(0)t+x_1(0)$$

Eliminate time variable t to gain:

$$x_1(t) = -\frac{x_2^2(t)}{2} + x_1(0) + \frac{x_2^2(0)}{2}$$

The optimal trajectory in phase plane can be described as the following expression:

$$x_1(t) = -\frac{x_2^2(t)}{2} + c$$

Above analysis demonstrates that the optimal trajectory of system is two bundles of parabola shown in figure7.1.

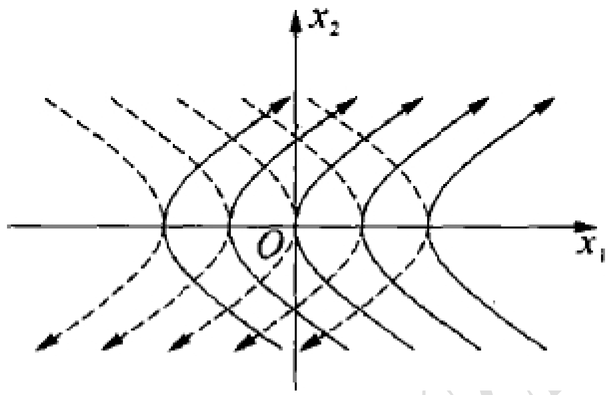

Figure7.1 Optimal trajectory bundle

The two curves through original point is the following:

$$x_1(t) = \frac{1}{2}x_2^2(t), \quad x_2(t) \le 0, \quad u(t) = 1$$

$$x_1(t) = -\frac{1}{2}x_2^2(t), \quad x_2(t) \ge 0, \quad u(t) = -1$$

Two curves of parabola through original point is called to switch curve which equation may be expressed as:

$$x_1(t) = -\frac{1}{2}x_2(t)|x_2(t)|$$

7.7 Linear Quadratic Optimal Control

If a control system is linear, and if performance functional is the quadratic function integration of state variable or control variable; such control problem is called to linear quadratic regulator problem, which is shortened to ~~LQR~~ (Linear Quadratic Regulator, LQR)problem. Such optimal control problem is very widely applied and an important result of modern control theory. Control law solved from linear quadratic problem is linear function of state variable; hence closed-loop optimal control can be realized through state feedback.

In this section, we firstly discuss linear quadratic functional, then regulator and follower.

7.7.1 Linear Quadratic Optimal Control Problem

Assume that linear system is as follows:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \tag{7.153}$$

The system state is completely controllable. Among equation(7.153), state vector $x \in R^n$; control vector $u \in R^p$; matrices A and B are respectively $n \times n$ and $n \times p$ matrices.

Performance index of optimal control is the linear quadratic function of state vector and control vector, which can be expressed as

$$\begin{aligned} J = & \frac{1}{2}\left[x(t_f) - x_d(t_f)\right]^T P\left[x(t_f) - x_d(t_f)\right] \\ & + \frac{1}{2}\int_{t_0}^{t_f} \left\{\left[x(t) - x_d(t)\right]^T Q(t)\left[x(t) - x_d(t)\right] + u^T(t)R(t)u(t)\right\}dt \end{aligned} \tag{7.154}$$

Among equation(7.154), desired system state $x_d(t)$, $t \in [t_0, \ t_f]$ represents desired state trajectory; weighting matrices P and Q is positive semidefinite matrix; R is positive symmetric matrix.

Above problem is called to linear quadratic optimal control problem.

Linear quadratic performance index in equation(7.154) has the following physical significance.

The first term represents the requirement of stable error. Weighting matrix P shows the attention degree on each state error.What we need to note is that stable error is smallest, not zero. Stable error should accord with the need of practical engineering. Coefficient 0.5 has no any physical and mathematical significance for deductive convenience.

The second term shows total tolerance of that actual state deviates from ideal state in the whole control course. Weighting matrix Q demonstrates the attention degree of each component of state vector. The third term reflects the cost of control, and represents the total control energy which is consumed in the whole control course.

LQR regulator problem means that control law designed may make system state return zero state in the proximity to satisfy that quadratic aim function is minimum.

In LQR problem, the specific means of realizing relative control is not pointed out clearly. Only a controller needs to be designed to make performance indexes reach minimum.

7.7.2 Finite time state regulator of linear continuous system

The task of state regulator is that system state keeps each components close to balancing state in the case of no overmuch energy cost when system state deviates from its stable state. When such problem is studied, original state vector is usually regarded as disturbance, and zero state is regarded as balancing state. Then regulator problem is converted into searching for optimal control law u in the limited time $\left[t_0,\ t_f\right]$ to make system be converted from original state to zero state in the proximity. Furthermore, regulator should make functional J get minimum.

We postulate that the state space of linear time-variant system is depicted as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t)+B(t)u(t) \\ y(t) &= C(t)x(t) \\ x(t_0) &= x_0 \end{aligned} \tag{7.155}$$

In above expression, x, u, y are respectively n, r, m dimension vectors. Time-variant matrix $A(t)$ is $n\times n$ system matrix; matrix $B(t)$ is $n\times r$ control matrix; matrix $C(t)$ is $m\times n$ output matrix.

Quadratic performance index functional of equation(7.155) is as follows:

$$J=\frac{1}{2}x(t_f)^T Px(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\left[x(t)^T Q(t)x(t)+u^T(t)R(t)u(t)\right]dt \tag{7.156}$$

Here, weighting matrix P is positive semidefinite and symmetric constant matrix; $Q(t)$ is positive semidefinite matrix, $R(t)$ is positive and symmetric matrix;each element in matrices $Q(t)$ and $R(t)$ is continuously boundary for time.

Above optimal problem is called to finite time regulator problem. We may employ minimum principle to solve it. Now let's construct Hamilton function:

$$H[x,\ u,\ \lambda,\ t]=\frac{1}{2}x(t)^T Q(t)x(t)+\frac{1}{2}u(t)^T R(t)u(t)+\lambda^T(t)[A(t)x(t)+B(t)u(t)] \tag{7.157}$$

Then we can get the following canonical equation:

$$\dot{x}(t)=\frac{\partial H}{\partial \lambda}=A(t)x(t)+B(t)u(t) \tag{7.158a}$$

$$\dot{\lambda}(t)=-\frac{\partial H}{\partial x}=-Q(t)x(t)-A^T(t)\lambda(t) \tag{7.158b}$$

For the control law $u(t)$ is free, the following minimum condition will be satisfied:

$$\frac{\partial H}{\partial u}=R(t)u(t)+B(t)^T\lambda(t)=0 \tag{7.159}$$

We can achieve the solution of above equation:

$$u^*(t)=-R^{-1}(t)B^T(t)\lambda(t) \tag{7.160}$$

The second derivative of Hamilton to control law is as follows:

$$\frac{\partial^2 H}{\partial u^2}=R(t)>0 \tag{7.161}$$

Hence, performance index functional J should get minimum. Equation(7.160) is replaced into canonical equation (7.158) and then gain:

$$\dot{x}(t)=A(t)x(t)-B(t)R^{-1}(t)B^T(t)\lambda(t) \tag{7.162}$$

$$\dot{\lambda}(t) = -Q(t)x(t) - A^T(t)\lambda(t) \tag{7.163}$$

Above equations are obviously one-second linear differential equation group. According to transverse condition which final state is free and which final time is given, its boundary condition and transverse condition is:

$$x(t_0) = x_0 \tag{7.164}$$

$$\lambda(t_f) = \frac{\partial\theta}{\partial x(t_f)} = \frac{\partial}{\partial x(t_f)}\left[\frac{1}{2}x^T(t_f)Px(t_f)\right] = Px(t_f) \tag{7.165}$$

Since final system state $x(t_f)$ and final adjoining vector have linear relationship in above transverse conditions, and furthermore since canonical equation is also linear, we can postulate that there is possibly the following linear relationship between system state $x(t)$ and adjoining vector $\lambda(t)$ at any time $t \in [t_0, \ t_f]$:

$$\lambda(t) = K(t)x(t) \tag{7.166}$$

Here, coefficient matrix $K(t)$ is undetermined $n \times n$ matrix. Equation(7.166) is differentiated and then we can get:

$$\dot{\lambda}(t) = \dot{K}(t)x(t) + K(t)\dot{x}(t) \tag{7.167}$$

Equation(7.162) and equation(7.166) are both replaced into equation(7.167), achieve:

$$\dot{\lambda}(t) = \left[\dot{K}(t) + K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t)\right]x(t) \tag{7.168}$$

Equation(7.166) is substituted into equation(7.163) and we can know:

$$\dot{\lambda}(t) = [-Q(t) - A^T(t)K(t)]x(t) \tag{7.169}$$

Order that equation(7.169) is equal to equation(7.168), we can achieve:

$$\dot{\lambda}(t) = \left[\dot{K}(t) + K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t)\right]x(t) = [-Q(t) - A^T(t)K(t)]x(t) \tag{7.170}$$

Above equality is always found for any system state $x(t)$. Hence, we have the following equation:

$$\dot{K}(t) + K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) = -Q(t) - A^T(t)K(t) \tag{7.171}$$

Namely,

$$\dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t) \tag{7.172}$$

Expression(7.172) is called to Riccati matrix differential equation which is a first-second nonlinear matrix differential equation in nature.

After comparing equation(7.165) with equation(7.166), we easily find that the boundary condition of equation(7.172) is the following:

$$K(t_f) = P \tag{7.173}$$

Matrix $K(t)$ which can satisfying equation(7.172) and equation(7.173) should be symmetric when $t \in [t_0, \quad t_f]$. Namely,

$$K(t) = K^T(t) \tag{7.174}$$

Equation(7.174) is proved as follows:

Equation(7.172) and equation(7.274) are transposed and gain:

$$\dot{K}^T(t) = K^T(t)B(t)R^{-1}(t)B^T(t)K^T(t) - A^T(t)K^T(t) - K^T(t)A(t) - Q(t)$$
$$K^T(t_f) = P$$

It is easily known that matrix $K(t)$ and matrix $K^T(t)$ are both the solution of equation(7.172) for same boundary condition; hence according to the uniqueness of solution in equation(7.172), we can easily know:

$$K^T(t) = K(t)$$

The undetermined matrix $K(t)$ is symmetric matrix.

After we gain the solution $K(t)$ from above Raccati equation(7.172), the optimal control law $u(t)$ may be achieved from equation(7.160):

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x(t) \tag{7.175}$$

Equation(7.175) is replaced into equation(7.155), and then we have

$$\dot{x}(t) = [A(t) - B(t)R^{-1}(t)B^T(t)K(t)]x(t), \qquad x(t_0) = x_0 \tag{7.176}$$

Obviously, the solution of above linear differential equation is the optimal solution $x^*(t)$ of system state.

Then, we firstly calculate and confirm the optimal value of performance index functional J.

Optimal control law $u^*(t)$ and optimal state $x^*(t)$ is substituted into performance index functional, and we can get that the minimum of performance index function is as follows:

$$J^* = \frac{1}{2}x^T(t_0)K(t_0)x(t_0) \tag{7.177}$$

Proof

Differentiate expression $x^T(t)K(t)x(t)$ and then we can get the following expression:

$$\frac{d}{dt}[x^T(t)K(t)x(t)] = \dot{x}^T(t)K(t)x(t) + x^T(t)\dot{K}(t)x(t) + x^T(t)K(t)\dot{x}(t)$$

Derivative $\dot{x}$ from system state equation is replaced into above equation, and $\dot{K}(t)$ is also substituted into above equation by Riccati matrix differential equation. And then we can gain:

$$\frac{d}{dt}[x^T Kx] = -x^T Qx - u^T Ru + [u + R^{-1}B^T Kx]^T R[u + R^{-1}B^T Kx]$$

When $u(t)$ and $x(t)$ are equal to optimal functions $u^*(t)$ and $x^*(t)$, have

$$\frac{d}{dt}\left[x^{*T}Kx^{*}\right]=-x^{*T}Qx^{*}-u^{*T}Ru^{*}$$

Integrate above equation from t_0 to t_f and multiply them by 0.5,

$$\frac{1}{2}\int_{t_0}^{t_f}\left\{\frac{d}{dt}\left[x^{*T}Kx^{*}\right]\right\}dt=-\frac{1}{2}\int_{t_0}^{t_f}\left[x^{*T}Qx^{*}+u^{*T}Ru^{*}\right]dt$$

Namely

$$\frac{1}{2}\left[x^{*T}Kx^{*}\right]_{t_0}^{t_f}=-\frac{1}{2}\int_{t_0}^{t_f}\left[x^{*T}Qx^{*}+u^{*T}Ru^{*}\right]dt$$

Above equation is replaced into equation(7.156), and then gain

$$J^{*}=J^{*}[x(t_0)]=\frac{1}{2}\int_{t_0}^{t_f}\left[x^{*T}Qx^{*}+u^{*T}Ru^{*}\right]dt+\frac{1}{2}x^{*T}(t_f)Kx^{*}(t_f)$$
$$=-\frac{1}{2}\left[x^{*T}Kx^{*}\right]_{t_0}^{t_f}+\frac{1}{2}x^{*T}(t_f)Kx^{*}(t_f)$$
$$=\frac{1}{2}x^{*T}(t_0)K(t_0)x^{*}(t_0)$$

Obviously, at any time, performance functional is

$$J^{*}=J^{*}[x(t)]=\frac{1}{2}x^{*T}(t)K(t)x^{*}(t)$$

When $t=t_f$,

$$J^{*}=J^{*}[x(t_f)]=\frac{1}{2}x^{*T}(t_f)P(t_f)x^{*}(t_f)=\frac{1}{2}x^{*T}(t_f)Px^{*}(t_f)$$

Theorem The given state equation of linear time-variant system is as follows:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t),\qquad x(t_0)=x_0$$

Quadratic performance index functional is the following:

$$J=\frac{1}{2}x(t_f)^{T}Px(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\left[x(t)^{T}Q(t)x(t)+u^{T}(t)R(t)u(t)\right]dt$$

In this equation, control variable u is not constrained; final time t_f is definite; and weighting matrices P and Q are positive semidefinite matrices; matrix R is positive symmetric matrix; then optimal control law $u^{*}(t)$ always exists and is unique, and furthermore it can be determined by the following formula:

$$u^{*}(t)=-R^{-1}(t)B^{T}(t)K(t)x(t)$$

Optimal system state $x^{*}(t)$ is the solution of the following equation:

$$\dot{x}(t)=\left[A(t)-B(t)R^{-1}(t)B^{T}(t)K(t)\right]x(t),\qquad x(t_0)=x_0$$

The minimum of performance index functional is

$$J^{*}=\frac{1}{2}x^{T}(t_0)K(t_0)x(t_0)$$

Several points need to be explained on above results:

1) Optimal control law is a linear state feedback, hence it's very convenient to realize closed-loop optimal control.

2) If control time period $\lfloor t_0, \quad t_f \rfloor$ is definite and if matrix $K(t)$ is time-variable, optimal control system is linear time-variant control system.

3) $K(t)$ is the solution of nonlinear differential equation; generally it is very difficult to solve the analytic solution. Hence, numerical solution of matrix $K(t)$ could be calculated by computer.

4) We should note that there is no controllable requirement of system in the finite time state regulator of linear continuous system.

Example7.13 A given state equation of linear time-constant system is as follows

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0$$

Performance index functional is

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)Ru(t)\right]dt$$

Relative matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1$$

Please find optimal control $u^*(t)$.

Solution:

We assume that positive and symmetric matrix is the following:

$$K(t) = \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix}$$

Above matrix could satisfy the following Riccati matrix differential equation:

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - Q(t)$$

$$\begin{bmatrix} \dot{k}_{11} & \dot{k}_{12} \\ \dot{k}_{21} & \dot{k}_{22} \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Namely,

$$\begin{bmatrix} \dot{k}_{11} & \dot{k}_{12} \\ \dot{k}_{21} & \dot{k}_{22} \end{bmatrix} = \begin{bmatrix} k_{12}^2 - 1 & -k_{11} + k_{12}k_{22} \\ -k_{11} + k_{12}k_{22} & -2k_{12} + k_{22}^2 \end{bmatrix}$$

We can get the following linear algebraic equation group:

$$\begin{cases} \dot{k}_{11} = k_{12}^2 - 1 \\ \dot{k}_{12} = -k_{11} + k_{12}k_{22} \\ \dot{k}_{22} = -2k_{12} + k_{22}^2 \end{cases}$$

Final boundary condition is $K(t_f)=P=0$. We utilize computer to solve above differential equation and gain the value of matrix $K(t)$ from $t=0$ to $t=t_f$. Then we can get optimal solution:

$$u^*(t)=-[0 \quad 1]\begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=-k_{12}(t)x_1(t)-k_{22}(t)x_2(t)$$

7.7.3 Infinite time-constant State regulator of linear continuous system

Assume the state equation of linear time-constant system is the following:

$$\dot{x}(t)=Ax(t)+Bu(t) \quad , \qquad x(t_0)=x_0 \tag{7.178}$$

In equation(7.178), matrices A and B are respectively $n\times n$ and $n\times p$ constant matrices.

Postulate that system $\sum(A, \quad B)$ is completely controllable, state vector $x\in R^n$, and that control vector $u(t)$ is not constrained. Quadratic performance index functional is

$$J=\frac{1}{2}\int_{t_0}^{\infty}\left[x^T(t)Qx(t)+u^T(t)Ru(t)\right]dt \tag{7.179}$$

Here, weighting matrices Q and R is constant matrix, Q is positive semidefinite , and R is positive and symmetric matrix. Control law $u(t)$ is free. Require that optimal control $u^*(t)$ needs to be confirmed to make performance index functional have minimum.

Comparing with previous state regulator with limited time, indefinite time-constant regulator has the following characteristics:

1) system is time-constant, and weighting matrix in performance index functional J is constant matrix;

2) Final time tends to infinity ∞. In infinite time time-constant regulator, that the final time t_f has a tendency toward infinity is to get a constant feedback matrix K.

3) Final weighting matrix $P=0$, namely there is no final performance requirement. This is because final performance will lose the engineering significance when final time t_f tends to infinity.

4) Requiring that system is completely controllable is to guarantee the stability of optimal system. When control area is infinite, system performance index tends possibly to infinity based on any control if system is not completely controllable. Then the advantages and disadvantages of control performance will not be compared to ensure the optimal control of system.

5) Closed-loop control system is asymptotically stable, namely all characteristic roots of system matrix $\left[A - BR^{-1}B^T K\right]$ have negative real.

Assume that Lyapunov's function is the following:

$$V(x) = x(t)^T Kx(t)$$

Lyapunov's function $V(x)$ is positive definite for matrix K is positive definite.

$$\dot{V}(x) = \dot{x}^T Kx + x^T K\dot{x}$$

Equation(7.176) is replaced into above equation, and then we can achieve:

$$\begin{aligned}\dot{V}(x) &= x^T(t)\left[A - BR^{-1}B^T K\right]^T Kx + x^T K\left[A - BR^{-1}B^T K\right]x \\ &= x^T\left[\left(A^T K + KA - KBR^{-1}B^T K\right) - KBR^{-1}B^T K\right]x \\ &= -x^T\left(Q + KBR^{-1}B^T K\right)x\end{aligned}$$

For matrix Q and matrix R is positive definite, derivative $\dot{V}(x)$ is negative definite.

Conclusion5) is proved. In practice, if $\dot{V}(x)$ is not eternally equal to zero, matrix Q may be positive semidefinite.

Theorem Assume linear time-constant system is the following:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \qquad x(t_0) = x_0$$

This system is completely controllable, and quadratic performance index functional is

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt$$

Here, weighting matrices Q and R is constant matrix, Q is positive semidefinite and symmetric matrix, and R is positive and symmetric matrix. Control law $u(t)$ is free.

Then optimal control law $u^*(t)$ always exists and is unique, which is decided by the following expression:

$$u^*(t) = -R^{-1}B^T Kx(t) \tag{7.180}$$

Here, matrix K is $n \times n$ positive symmetric constant matrix, it is the unique solution of the following Riccati matrix algebraic equation:

$$KA + A^T K - KBR^{-1}B^T K + Q = 0 \tag{7.181}$$

Optimal state $x^*(t)$ is the solution of the following differential equation:

$$\dot{x}(t) = \left[A - BR^{-1}B^T K\right]x(t), \quad x(t_0) = x_0 \tag{7.182}$$

The minimum of performance index functional is as follows:

$$J^* = \frac{1}{2}x^T(t_0)Kx(t_0) \tag{7.183}$$

Example7.14 A given state equation of linear time-constant continuous system is as follows

$$\dot{x}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

Performance index functional is

$$J=\frac{1}{2}\int_0^{\infty}\left(x_1^2+2bx_1x_2+ax_2^2+u\right)dt$$

Please find optimal control $u^*(t)$ when functional J gets minimum.

Solution: In terms of performance index functional and system state equation,the following matrices may be known:

$$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix},\quad Q=\begin{bmatrix}1 & b\\ b & a\end{bmatrix},\quad R=1$$

In order to make that matrix Q is positive definite, the following inequality needs to be always found:

$$a-b^2>0$$

Judge whether given system is completely controllable.

$$rankS_c=rank\begin{bmatrix}B & AB\end{bmatrix}=rank\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=2$$

Hence, given system is completely controllable. Optimal control $u^*(t)$ always exists for matrix Q and matrix R is positive definite.

$$u^*(t)=-R^{-1}B^TKx(t)=-1\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=-k_{12}x_1(t)-k_{22}x_2(t)$$

k_{21} and k_{22} is the solution of the following Riccati differential equation:

$$KA+A^TK-KBR^{-1}B^TK+Q=0$$

$$\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}-\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}+\begin{bmatrix}1 & b\\ b & a\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Namely,

$$\begin{bmatrix}-k_{12}^2+1 & k_{11}-k_{12}k_{22}+b\\ k_{11}-k_{12}k_{22}+b & 2k_{12}-k_{22}^2+a\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Develop and then gain the following algebraic equation group:

$$-k_{12}^2+1=0$$

$$k_{11}-k_{12}k_{22}+b=0$$

$$2k_{12}-k_{22}^2+a=0$$

The solution of above equation group is as follows:

$$k_{12}=\pm 1,\quad k_{22}=\pm\sqrt{2k_{12}+a},\quad k_{11}=k_{12}k_{22}-b$$

After considering positive definite matrix K, we can get:

$$k_{12}=1,\quad k_{22}=\sqrt{2+a},\quad k_{11}=\sqrt{a+2}-b$$

$$K = \begin{bmatrix} \sqrt{a+2}-b & 1 \\ 1 & \sqrt{a+2} \end{bmatrix}$$

Hence, optimal control $u^*(t)$ can be gained:

$$u^*(t) = -R^{-1}B^T Kx(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{a+2}-b & 1 \\ 1 & \sqrt{a+2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -x_1(t) - \sqrt{a+2}x_2(t)$$

The state equation of closed-loop system is the following:

System matrix:

$$A - BR^{-1}B^T K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{a+2}-b & 1 \\ 1 & \sqrt{a+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{a+2} \end{bmatrix}$$

State equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{a+2} \end{bmatrix} x$$

State feedback transfer function of closed-loop system:

$$G(s) = C[sE - (A + BK)]^{-1} B = \frac{1}{s^2 + s\sqrt{a+2} + 1}$$

Poles of closed-loop system are as follows:

$$s_{1,2} = -\frac{\sqrt{a+2}}{2} \pm j\frac{\sqrt{2-a}}{2} \qquad \text{(when } a < 2\text{)}$$

7.7.4 Output regulator of linear continuous system

In the above sections, we have discussed the design problem of state regulator. The performance index functional of state regulator requires that weighting square sum of all states of that system deviates from balancing point is minimum. However, plenty of optimal control problems in practice do not emphasize all states of system but only have the requirement on the output of control system. When people utilizes state equation to describe a physical system, they difficultly give out specific requirement on all system states; furthermore a lot of system variables don't have any physical significance possibly. But people can put forward specific requirement on system output. Hence, it's very significant to put forward relative requirement on system output to construct the performance index functional of optimal control system.

In this section, we will solve output regulator problem of linear continuous system.

1. Finite time output regulator

We postulate that linear time-variant system is as follows:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0 \tag{7.184a}$$

$$y = C(t)x(t) \tag{7.184b}$$

The performance index functional for finite time output regulator is the following:

$$J = \frac{1}{2} y^T(t_f) P y(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[y^T(t) Q(t) y(t) + u^T(t) R(t) u(t) \right] dt \qquad (7.185)$$

In above equation, control law $u(t)$ is not limited and final time t_f is given; time-variant matrices $A(t)$, $B(t)$, $C(t)$ are continuous boundary function on time variable; weighting matrix P is positive semidefinite and symmetric constant matrix; weighting matrices $Q(t)$, $R(t)$ are respectively positive semidefinite and positive definite symmetric matrices which each element is continuously boundary on time variable.

Please find optimal control $u^*(t)$ make that performance index functional J has minimum.

In order to solve such problem, we should firstly transform it into equivalent problem of state regulator; and then we may utilize the previous result of state regulator to solve optimal control law $u^*(t)$.

Expression in equation(7.184b) is replaced into equation(7.185), and then we achieve:

$$\begin{aligned} J &= \frac{1}{2} y^T(t_f) P y(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[y^T(t) Q(t) y(t) + u^T(t) R(t) u(t) \right] dt \\ &= \frac{1}{2} x^T(t_f) C^T(t_f) P C(t_f) x(t) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t) C^T(t) Q(t) C(t) x(t) + u^T(t) R(t) u(t) \right] dt \end{aligned} \qquad (7.185)$$

Comparing with equation(7.156), we may find that weighting matrices P and $Q(t)$ have been changed. New weighting matrices are marked as:

$$\overline{P} = C^T(t_f) P C(t_f), \qquad \overline{Q}(t) = C^T(t) Q(t) C(t) \qquad (7.186)$$

Then relative performance index functional J is the following:

$$J = \frac{1}{2} x^T(t_f) \overline{P} x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t) \overline{Q}(t) x(t) + u^T(t) R(t) u(t) \right] dt \qquad (7.187)$$

After comparing equation(7.187) with equation(7.156), we easily know that if matrices $\overline{P}$ and $\overline{Q}(t)$ are positive semidefinite matrix, the method of solving state regulator may be utilized to solve optimal control $u^*(t)$ of output regulator.

If matrices P and $Q(t)$ are symmetric matrix, matrices $\overline{P}$ and $\overline{Q}(t)$ are also symmetric.If system is observable, matrix $C^T(t)$ is not zero at $t \in [t_0, \ t_f]$. If matrix Q is positive semidefinite, performance term $y^T(t) Q y(t)$ is always greater than or equal to zero, namely $y(t) Q y(t) \ge 0$ for all system output $C(t)x(t)$. For system output is completely observable, each output is formed by unique system state $x(t)$. Hence we may think that matrix $C^T(t) Q C(t)$ is positive semidefinite. Similarly, we may deduce that matrix $C^T(t) P C(t)$ is also positive semidefinite.

Then from optimal control of state regulator, we may easily confirm the optimal control $u^*(t)$:

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x(t)$$

Theorem When and only when linear time-variant system in equation(7.184) is completely observable, there are uniquely optimal control:

$$u^*(t) = -R^{-1}(t)B^T(t)K(t)x(t) \tag{7.188}$$

Here, gain matrix $K(t)$ is positive definite symmetric matrix, and it is unique solution of the following Riccati differential equation:

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - C^T(t)Q(t)C(t) \tag{7.189}$$

$$K(t_f) = C^T(t_f)PC(t_f)$$

Optimal state $x^*(t)$ is the solution of the following differential equation:

$$\dot{x}(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x(t), \qquad x(t_0) = x_0 \tag{7.190}$$

Optimal performance index functional is as follows:

$$J^* = \frac{1}{2}x^T(t_0)K(t_0)x(t_0) \tag{7.191}$$

From equation(7.188), we easily find that optimal output regulator is still state feedback, not output feedback. Optimal control law is the same as that of optimal state regulator. Their difference is only that they have different Riccati equations. This is because system observability ensures the state of system output estimation state. However, system output only shows the linear combination of each state component, and it can not provide the all information of each state component, but optimal control needs the information of all state variables. Hence, optimal output regulator is still state feedback, not output feedback.

2.Infinite time output regulator

Postulate that linear time-constant system is as follows:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{7.192a}$$

$$y = Cx(t) \tag{7.192b}$$

For infinite time output regulator, quadratic performance index functional yields to

$$J = \frac{1}{2}\int_0^{\infty}\left[y^T(t)Qy(t) + u^T(t)Ru(t)\right]dt \tag{7.193}$$

Here, weighting matrices Q and R are respectively positive semidefinite and positive definite symmetric matrices. Please find optimal control $u^*(t)$ to make that performance index functional J is minimum.

The deducing course is similar to that in previous theme. We may utilize the result of infinite time state regulator to gain the result of infinite time output regulator.

Theorem When and only when linear time-constant system in equation(7.192) is completely controllable and completely observable, there is an uniquely optimal control $u^*(t)$:

$$u^*(t) = -R^{-1}B^T Kx(t) \quad (7.194)$$

Here, gain matrix K is positive definite symmetric matrix, and it is the unique solution of the following Riccati algebraic equation:

$$KA + A^T K - KBR^{-1}B^T K + C^T QC = 0 \quad (7.195)$$

Optimal state control $x^*(t)$ is the solution of the following differential equation:

$$\dot{x}(t) = \left[A - BR^{-1}B^T K\right]x(t), \quad x(t_0) = x_0 \quad (7.196)$$

Optimal performance index functional is as follows:

$$J^* = \frac{1}{2}x^T(t_0)Kx(t_0) \quad (7.197)$$

Example7.15 A given state equation of linear time-constant continuous system is as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(t_0) = x_0$$

Output equation:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Performance index functional is

$$J = \frac{1}{2}\int_0^{\infty} \left[y^2(t) + qu^2(t)\right]dt$$

Please find optimal control $u^*(t)$ when functional J gets minimum.

Solution:

1) Judge whether given system is completely observable and completely controllable.

$$rankS_c = rank\begin{bmatrix} B & AB \end{bmatrix} = rank\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2$$

$$rankS_o = rank\begin{bmatrix} C \\ CA \end{bmatrix} = rank\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

Hence, given system is completely controllable and completely observable.

2) Ascertain the relative matrices in Riccati algebraic equation.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q = 1, \quad R = q$$

3) Acquire gain matrix K from Riccati algebraic equation.

Assume the positive definite symmetric matrix:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

$$C^T QC = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Gain matrix K can satisfy the following Riccati algebraic equation:

$$KA + A^T K - KBR^{-1}B^T K + C^T QC = 0$$

Namely,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} q^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Simplify and merge:

$$\begin{bmatrix} -\frac{1}{q}k_{12}^2 + 1 & k_{11} - \frac{1}{q}k_{12}k_{22} \\ k_{11} - \frac{1}{q}k_{12}k_{22} & 2k_{12} - \frac{1}{q}k_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, the following algebraic equation can be achieved from above matrix equation:

$$\begin{cases} -\frac{1}{q}k_{12}^2 + 1 = 0 \\ k_{11} - \frac{1}{q}k_{12}k_{22} = 0 \\ 2k_{12} - \frac{1}{q}k_{22}^2 = 0 \end{cases}$$

The solution of above algebraic equation group is gained as follows:

$$k_{11} = \pm\sqrt{2\sqrt{q}}, \qquad k_{12} = \sqrt{q}, \qquad k_{22} = \pm\sqrt{2q\sqrt{q}} \qquad (\text{when } q > 0)$$

Since gain matrix K is positive definite, its solution is ensured as

$$k_{11} = \sqrt{2\sqrt{q}}, \qquad k_{12} = \sqrt{q}, \qquad k_{22} = \sqrt{2q\sqrt{q}} \qquad (\text{when } q > 0)$$

Hence, gain matrix K can be written out in the following:

$$K = \begin{bmatrix} \sqrt{2\sqrt{q}} & \sqrt{q} \\ \sqrt{q} & \sqrt{2q\sqrt{q}} \end{bmatrix}$$

4) Respectively give out optimal control law $u^*(t)$ and optimal state $x^*(t)$ and optimal performance index functional:

Optimal control law $u^*(t)$:

$$u^*(t) = -R^{-1}B^T Kx(t) = -\frac{1}{q}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2\sqrt{q}} & \sqrt{q} \\ \sqrt{q} & \sqrt{2q\sqrt{q}} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\frac{\sqrt{q}}{q}x_1(t) - \frac{\sqrt{2q\sqrt{q}}}{q}x_2(t)$$

Solve the optimal state $x^*(t)$ from state equation:

$$A - BR^{-1}B^T K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{q}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2\sqrt{q}} & \sqrt{q} \\ \sqrt{q} & \sqrt{2q\sqrt{q}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{q}}{q} & -\frac{\sqrt{2q\sqrt{q}}}{q} \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{q}}{q} & -\frac{\sqrt{2q\sqrt{q}}}{q} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x^*(t)=x(t_0)\exp\left\{\begin{bmatrix}0 & 1\\ -\frac{\sqrt{q}}{q} & -\frac{\sqrt{2q\sqrt{q}}}{q}\end{bmatrix}(t-t_0)\right\}$$

Optimal performance index functional J^* from equation(7.197):

$$J^*=\frac{1}{2}x^T(t_0)Kx(t_0)=\frac{1}{2}[x_1(t_0)\quad x_2(t_0)]\begin{bmatrix}\sqrt{2\sqrt{q}} & \sqrt{q}\\ \sqrt{q} & \sqrt{2q\sqrt{q}}\end{bmatrix}\begin{bmatrix}x_1(t_0)\\ x_2(t_0)\end{bmatrix}$$

$$=\frac{1}{2}\left[\sqrt{2\sqrt{q}}x_1^2(t_0)+2\sqrt{q}x_1(t_0)x_2(t_0)+\sqrt{2q\sqrt{q}}x_2^2(t_0)\right]$$

7.7.5 Output follower of linear continuous system

Control objective of output follower is to make system output $y(t)$ tightly follow the desired output $z(t)$ without consuming plenty of control energy.

1. Follower problem of linear time-variant system

Give out a linear time-variant system as follows:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t) \quad (7.198a)$$

$$y(t)=C(t)x(t) \quad (7.198b)$$

$$x(t_0)=x_0 \quad (7.198c)$$

Postulate that control law $u(t)$ is not constrained. Sign vector $z(t)$ expresses the desired system output, which has the same dimension as that of actual system output $y(t)$. The error vector $e(t)$ is defined as the following:

$$e(t)=z(t)-y(t) \quad (7.199)$$

Or, written into another form:

$$e(t)=z(t)-C(t)x(t) \quad (7.200)$$

Find the optimal control $u^*(t)$ to make the following performance index functional J have minimum.

$$J=\frac{1}{2}e^T(t_f)Pe(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\left[e^T(t)Q(t)e(t)+u^T(t)R(t)u(t)\right]dt \quad (7.201)$$

Here, weighting matrices $Q(t)$, $R(t)$ are respectively positive semidefinite and positive definite symmetric matrices; weighting constant matrix P is positive semidefinite symmetric matrix.

In the following paragraphs, minimum principle will be employed to solve the optimal problem. And construct Hamilton function as the following form:

$$H(x,\ u,\ \lambda,\ t)=\frac{1}{2}e^T(t_f)Q(t)e(t_f)+\frac{1}{2}e^T(t_f)R(t)e(t_f)+\lambda^T(t)[A(t)x(t)+B(t)u(t)] \quad (7.202)$$

Then canonical equation is as follows:

$$\dot{x}(t)=\frac{\partial H}{\partial \lambda}=A(t)x(t)+B(t)u(t) \tag{7.203a}$$

$$\begin{aligned}\dot{\lambda}(t)&=-\frac{\partial H}{\partial x}=-C^T(t)Q(t)C(t)x(t)-A^T(t)\lambda(t)+C^T(t)Q(t)z(t)\\&=C^T(t)Q(t)[z(t)-C(t)x(t)]-A^T(t)\lambda(t)\end{aligned} \tag{7.203b}$$

The following equation can be gained from minimum condition of Hamilton equation for control law is not constrained:

$$\frac{\partial H}{\partial u}=R(t)u(t)+B^T(t)\lambda(t)=0 \tag{7.204a}$$

Namely,

$$u^*(t)=-R^{-1}(t)B^T(t)\lambda(t) \tag{7.204b}$$

For matrix $R(t)$ is positive definite, above control law $u^*(t)$ may make Hamilton function have minimum.

Boundary condition:

$$x(t_0)=x_0 \tag{7.205a}$$

Transverse condition:

$$\begin{aligned}\lambda(t_f)&=\frac{1}{2}\frac{\partial}{\partial x(t_f)}\left\{[z(t_f)-C(t_f)x(t_f)]^T P[z(t_f)-C(t_f)x(t_f)]\right\}\\&=C^T(t_f)PC(t_f)x(t_f)-C^T(t_f)Pz(t_f)\\&=-C^T(t_f)P[z(t_f)-C(t_f)x(t_f)]\\&=-C^T(t_f)Pe(t_f)\end{aligned} \tag{7.205b}$$

From equation(7.203) and equation(7.204b), we can write out the following canonical equation:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{bmatrix}=\begin{bmatrix}A(t) & -B(t)R^{-1}B^T(t)\\ -C^T(t)Q(t)C(t) & -A^T(t)\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}+\begin{bmatrix}0\\ C^T(t)Q(t)\end{bmatrix}z(t) \tag{7.206}$$

The solution of above canonical equation is as follows:

$$\begin{aligned}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}&=\exp\left\{\begin{bmatrix}A(t) & -B(t)R^{-1}B^T(t)\\ -C^T(t)Q(t)C(t) & -A^T(t)\end{bmatrix}(t-t_0)\right\}\begin{bmatrix}x(t_0)\\ \lambda(t_0)\end{bmatrix}\\&+\int_{t_0}^{t}\exp\left\{\begin{bmatrix}A(t) & -B(t)R^{-1}B^T(t)\\ -C^T(t)Q(t)C(t) & -A^T(t)\end{bmatrix}(t-\tau)\right\}\begin{bmatrix}0\\ C^T(t)Q(t)\end{bmatrix}z(\tau)d\tau\end{aligned} \tag{7.207}$$

Canonical equation(7.203) is linear, and $\lambda(t_f)$, $x(t_f)$ and $z(t_f)$ among transverse condition have linear relationship; hence we may postulate the following relationship:

$$\lambda(t)=K(t)x(t)-g(t) \tag{7.208}$$

Here, matrix $K(t)$ is $n\times n$ undetermined matrix; and function $g(t)$ is unknown function which is related to the desired function $z(t)$.

Equation(7.208) is differentiated and the following expression can be achieved:

$$\dot{\lambda}(t) = \dot{K}(t)x(t) + K(t)\dot{x}(t) - \dot{g}(t) \tag{7.209}$$

Equation(7.204b) and equation(7.208) are replaced into equation(7.203a), and then we can get:

$$\dot{x}(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x(t) + B(t)R^{-1}(t)B^T(t)g(t) \tag{7.210}$$

Above equation(7.210) is substituted into equation(7.209); and then the following expression can be gained:

$$\dot{\lambda}(t) = \left[\dot{K}(t) + K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t)\right]x(t) + B(t)R^{-1}(t)B^T(t)g(t) - \dot{g}(t) \tag{7.211}$$

Equation(7.208) is replaced into equation(7.203b); and then there is the following expression:

$$\begin{aligned}\dot{\lambda}(t) &= C^T(t)Q(t)[z(t) - C(t)x(t)] - A^T(t)[K(t)x(t) - g(t)] \\ &= \left[-C^T(t)Q(t)C(t) - A^T(t)K(t)\right]x(t) + C^T(t)Q(t)z(t) + A^T(t)g(t)\end{aligned} \tag{7.209}$$

Comparing equation(7.209) with equation(7.211), we may find that the following expression should be always found:

$$\begin{aligned}&\left[\dot{K}(t) + K(t)A(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t)\right]x(t) + B(t)R^{-1}(t)B^T(t)g(t) - \dot{g}(t) \\ &= \left[-C^T(t)Q(t)C(t) - A^T(t)K(t)\right]x(t) + C^T(t)Q(t)z(t) + A^T(t)g(t)\end{aligned} \tag{7.210}$$

Equality (7.210) should be always found for any system state $x(t)$ and desired system output $z(t)$; hence the corresponding terms in both sides of above equality(7.210), and then they yields to the following two equations:

$$\dot{K}(t) = -K(t)A(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - A^T(t)K(t) - C^T(t)Q(t)C(t) \tag{7.211}$$

$$\dot{g}(t) = \left[B(t)R^{-1}(t)B^T(t) - A^T(t)\right]g(t) - C^T(t)Q(t)z(t) \tag{7.212}$$

Comparing equation(7.208) with equation(7.205b), we find that the boundary condition of above differential equations can be acquired:

$$K(t_f) = C^T(t_f)PC(t_f) \tag{7.213}$$

$$g(t_f) = C^T(t_f)Pz(t_f) \tag{7.214}$$

We may utilize computer software to solve differential equation from equation(7.211) to equation(7.212) by inverse time sequence, and then we acquire functions $K(t)$ and $g(t)$. And they are substituted into equation(7.208) and furthermore equation(7.204b) is utilized. Finally we can confirm that optimal control law $u^*(t)$ is as follows:

$$u^*(t) = -R^{-1}(t)B^T(t)[K(t)x(t) - g(t)] \tag{7.215}$$

Optimal system state may be solved from equation(7.210).And optimal performance index functional J^* is as follows:

$$J^* = \frac{1}{2}x^T(t_0)K(t_0)x(t_0) - g^T(t_0)x(t_0) + \phi(t_0) \tag{7.216}$$

Here, function $\phi(t_0)$ has to satisfy the following differential equation and boundary condition:

$$\dot{\phi}(t) = -\frac{1}{2}B^T(t)Q(t)z(t) - g^T(t)B(t)R^{-1}(t)B^T(t)g(t) \tag{7.217}$$

$$\phi(t_f) = z^T(t_f)K(t_f)z(t_f) \tag{7.218}$$

After comparing equation(7.211), equation(7.213) with equation(7.189), we can easily find that they are completely same. This shows that there is completely same feedback structure between feedback structure of optimal follower and feedback structure of optimal regulator. Optimal follower is not related to desired system output $z(t)$. By comparing equation(7.210) with equation(7.190), it is easily seen that characteristic roots of closed-loop control system between optimal follower closed-loop system and optimal output regulator are completely equal and that they are not related to the changing law of desired output $z(t)$. The difference between them only lies in that an input term is added in the follower system which is related to function $g(t)$.

2. Follower problem of linear time-invariant system

In above content, we have discussed the follower problem of linear time-variant system at $t \in [t_0,\ t_f]$. However, linear time-invariant system is very common and practical in engineering.

For a controllable and observable system, the state equation can be depicted as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7.219a}$$

$$y(t) = C(t)x(t) \tag{7.219b}$$

$$x(t_0) = x_0 \tag{7.219c}$$

Assume that the desired system output $z(t)$ is a constant vector, and then error function $e(t)$ can be expressed as follows:

$$e(t) = z - y(t) = z - Cx(t) \tag{7.220}$$

Performance index functional J can be depicted as

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[e^T(t)Qe(t) + u^T(t)Ru(t)\right]dt \tag{7.221}$$

Here, matrix Q and matrix R are both positive definite matrices.

When final time t_f is given, the following results by imitating previous results can be acquired:

$$u^*(t) = -R^{-1}B^T\left[Kx(t) - g(t)\right] \tag{7.222}$$

Among equation(7.222), matrix K and function $g(t)$ satisfy the following equation:

$$-KA + KBR^{-1}B^TK - A^TK - C^TQC = 0 \tag{7.223}$$

$$\dot{g}(t)=\left(KBR^{-1}B^T-A^T\right)g(t)-C^TQz \quad (7.224)$$

Example7.16 First-order dynamic system is depicted as follows

$$\dot{x}(t)=ax(t)+u(t)$$

Output equation: $y(t)=x(t)$

Control law $u(t)$ is not constrained. And the desired system output is expressed as sign $z(t)$; and error expression can be depicted as the following equality:

$$e(t)=z(t)-y(t)=z(t)-x(t)$$

Performance index functional J is

$$J=\frac{1}{2}pe^2(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\left[qe^2(t)+ru^2(t)\right]dt$$

Here, $p>0$, $q>0$, $r>0$. Please find optimal control $u^*(t)$ when functional J gets minimum.

Solution:

In terms of equation(7.222), we can get the optimal control law $u^*(t)$:

$$u^*(t)=-R^{-1}B^T\left[Kx(t)-g(t)\right]=-\frac{1}{r}\left[Kx(t)-g(t)\right]$$

Gain K is the solution of the following Riccati algebraic equation:

$$\dot{K}=-aK+\frac{1}{r}K^2-aK-q\,,\quad K(t_f)=p$$

The solution of above algebraic equation:

$$K(t)=r\frac{a+\beta-\dfrac{\dfrac{p}{r}-a-\beta}{\dfrac{p}{r}-a+\beta}(a-\beta)\exp\left[2\beta(t-t_f)\right]}{1-\dfrac{\dfrac{p}{r}-a-\beta}{\dfrac{p}{r}-a+\beta}(a-\beta)\exp\left[2\beta(t-t_f)\right]}$$

Here, $\beta=\sqrt{a^2+\dfrac{q}{r}}$

In terms of equation(7.212) and equation(7.214), the differential equation and boundary condition are:

$$\dot{g}(t)=-\left[a-\frac{K(t)}{r}\right]g(t)-qz(t),\qquad g(t_f)=pz(t_f)$$

Optimal track curve of system is the solution of the following differential equation:

$$\dot{x}(t)=\left[a-\frac{1}{r}k(t)\right]x(t)+\frac{1}{r}g(t),\quad x(0)=x_0$$

Example7.17 A given state equation of linear time-constant continuous system is as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u , \quad x(t_0) = x_0$$

System output equation:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Performance index functional J is

$$J = \frac{1}{2}\int_0^{t_f} \left\{ [x(t) - z(t)]^2 + u^2(t) \right\} dt$$

Please find optimal control $u^*(t)$ when functional J gets minimum.(assume that desired system output z is r, namely, $z(t) = r$)

Solution:

In terms of given information in this example, the following matrices can be known:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q = 1, \quad R = 1, \quad P = 0$$

According to equation(7.223), the matrix equation can be written out:

$$-KA + KBR^{-1}B^T K - A^T K - C^T QC = 0$$

Namely,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = 0$$

Simplify and merge above matrix equation; then we can gain:

$$\begin{bmatrix} 1 - k_{12}^2 & k_{11} - k_{12}k_{22} \\ k_{11} - k_{12}k_{22} & 2k_{12} - k_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then the following algebraic equation group can be gained from above equation:

$$\begin{cases} 1 - k_{12}^2 = 0 \\ k_{11} - k_{12}k_{22} = 0 \\ 2k_{12} - k_{22}^2 = 0 \end{cases}$$

Consider the characteristic of gain matrix K, we can get the following solution:

$$k_{11} = k_{22} = \sqrt{2} , \quad k_{12} = 1$$

Hence, gain matrix K may be written out as follows:

$$K = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

Now let's find the function $g(t)$. This function $g(t)$ can be expressed in terms of equation(7.224):

$$\dot{g}(t) = \left(KBR^{-1}B^T - A^T\right)g(t) - C^T Qz$$

Namely,

$$\dot{g}(t)=\left\{\begin{bmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}-\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right\}g(t)-\begin{bmatrix}1\\0\end{bmatrix}r$$

Simplify and merge, then get:

$$\dot{g}(t)=\begin{bmatrix}0 & 1\\ -1 & \sqrt{2}\end{bmatrix}g(t)-\begin{bmatrix}1\\0\end{bmatrix}r$$

Final condition of function $g(t)$ is: $g(t_f)=0$.

Utilizing integral method is to solve the expression of function $g(t)$.

$$g(t)=e^{\begin{bmatrix}0 & 1\\ -1 & \sqrt{2}\end{bmatrix}(t-t_0)}g(t_0)+\int_{t_0}^{t}e^{\begin{bmatrix}0 & 1\\ -1 & \sqrt{2}\end{bmatrix}(t-\tau)}\begin{bmatrix}-1\\0\end{bmatrix}r\,d\tau$$

When $t_0=0$, above expression is simplified into the following form:

$$g(t)=e^{\begin{bmatrix}0 & 1\\ -1 & \sqrt{2}\end{bmatrix}t}g(0)+\int_{t_0}^{t}e^{\begin{bmatrix}0 & 1\\ -1 & \sqrt{2}\end{bmatrix}(t-\tau)}\begin{bmatrix}-1\\0\end{bmatrix}r\,d\tau$$

Hence, optimal control $u^*(t)$ can be gained:

$$u^*(t)=-R^{-1}B^T[Kx(t)-g(t)]=-\begin{bmatrix}0 & 1\end{bmatrix}\left\{\begin{bmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}-\begin{bmatrix}g_1(t)\\g_2(t)\end{bmatrix}\right\}=-x_1(t)-\sqrt{2}x_2(t)+g_2(t)$$

7.8 Linear Quadratic Sub-optimal Control

In previous sections, state regulator,output regulator and output follower have been discussed to search relatively optimal control $u^*(t)$ which is realized by the linear combination of all system state variables $x_1(t),\ x_2(t),\ \cdots,\ x_n(t)$. However, in actual engineering,it's very difficult to acquire all information of system state variables. Then we utilize system outputs $y_1(t),\ y_2(t),\ \cdots,\ y_l(t)$ to form the control law $u(t)$ by combination. That's to say, system outputs $y_1(t),\ y_2(t),\ \cdots,\ y_l(t)$ with lower dimension are employed to form output feedback system. Then the performance index functional of such system with no complete information is worse than that of optimal control system. Such control is rendered to **sub-optimal control**.

For sub-optimal control, input control $u(t)$ is expressed through system output:

$$u(t)=-Ky(t) \tag{7.225}$$

This equation is output feedback in nature.

Assume the the completely controllable and observable system is given as follows:

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)\end{aligned} \tag{7.226}$$

Performance index functional of sub-optimal control system is as follows:

$$J = \frac{1}{2}\int_0^{\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt \qquad (7.227)$$

Here, matrices Q and R are positive definite matrices.

The block diagram of such closed-loop control system is shown in figure7.2

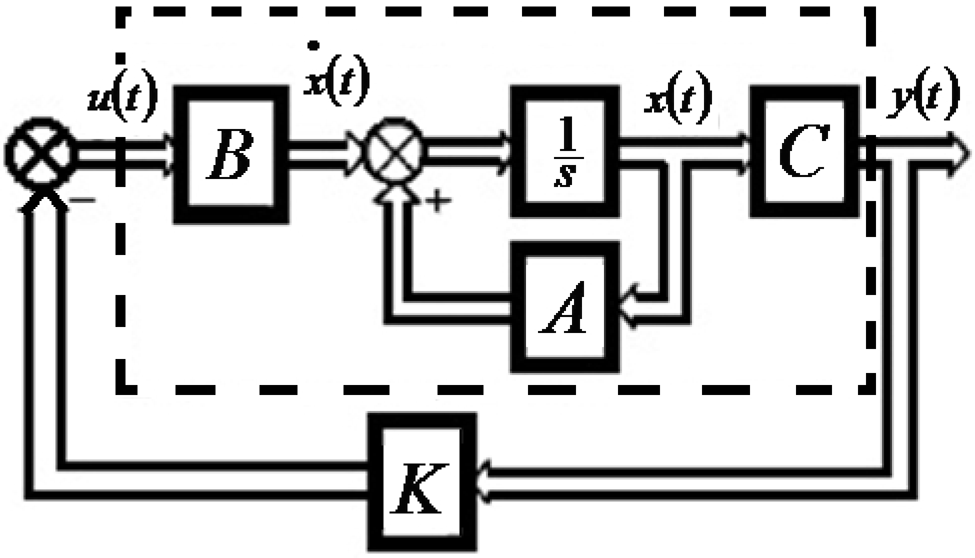

Figure7.2 Output feedback block diagram of closed-loop control system

From figure7.2, the state equation of closed-loop control system is gained:

$$\dot{x}(t) = [A - BKC]x(t) = \overline{A}x(t) \qquad (7.228)$$

Here, matrix $\overline{A} = A - BKC$ is the state matrix of closed-loop control system in figure7.2. Then, performance index functional J may be deduced into the following form:

$$\begin{aligned} J &= \frac{1}{2}\int_0^{\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt \\ &= \frac{1}{2}\int_0^{\infty}\left[x^T(t)Qx(t) + x^T(t)C^TK^TRKCx(t)\right]dt \\ &= \frac{1}{2}\int_0^{\infty}x^T(t)\left[Q + C^TK^TRKC\right]x(t)dt \\ &= \frac{1}{2}\int_0^{\infty}x^T(t)\overline{Q}x(t)dt \end{aligned} \qquad (7.229)$$

Here, $\overline{Q} = Q + C^TK^TRKC$. (7.230)

In this case of figure7.2, the task of design is to ensure the output feedback matrix K to make performance index functional J in equation(7.230) get minimum.

Lyapunov second method may be utilized to solve such problem. Firstly, all characteristic roots of closed-loop state matrix $\overline{A}$ both have negative reals, then closed-loop system is asymptotically stable. Secondly, utilize the relationship between the second method of Lyapunov function and quadratic performance index functional J.

For asymptotic system in equation(7.228), we construct Lyapunov function:

$$V(x) = x^T(t)Px(t). \qquad (7.230)$$

Matrix P is a symmetric matrix with all reals. First-order derivative is exerted into both sides of equation(7.230); and then we can gain:

$$\dot{V}(x)=\dot{x}^T(t)Px(t)+x^T(t)P\dot{x}(t)=x^T(t)\overline{A}^T Px(t)+x^T(t)P\overline{A}x(t)=x^T(t)\left(\overline{A}^T P+P\overline{A}\right)x(t) \quad (7.231)$$

For asymptotic system, if Lyapunov function $V(x)$ is positive definite, its derivative $\dot{V}(x)$ has to be negative definite; and then order:

$$\overline{A}^T P+P\overline{A}=-\overline{Q} \quad (7.231)$$

Here, matrix Q is positive definite real symmetric matrix.

Then equation(7.231) can be converted into the following form:

$$\dot{V}(x)=-x^T(t)\overline{Q}x(t) \quad (7.232)$$

Hence, function $\dot{V}(x)$ is negative definite. Comparing equation(7.232) with equation(7.230), we easily get:

$$x^T(t)\overline{Q}x(t)=-\frac{d}{dt}\left[x^T(t)Px(t)\right] \quad (7.233)$$

Hence, an asymptotic system has to satisfy equation(7.231) and equation(7.233), here, matrices $\overline{Q}$ and P have to be both positive definite and real symmetric matrices.

Equation(7.233) is substituted into equation(7.229), and then we can achieve:

$$J=\frac{1}{2}\int_0^\infty x^T(t)\overline{Q}x(t)dt=-\frac{1}{2}\left[x^T(t)Px(t)\right]_0^\infty=-\frac{1}{2}x^T(\infty)Px(\infty)+\frac{1}{2}x^T(0)Px(0) \quad (7.234)$$

Characteristic matrix $\overline{A}$ both has negative reals for all eigenvalue. Hence, there is

$$\lim_{t\to\infty}x(t)=0 \quad (7.235)$$

Then the following expression is found:

$$J=\frac{1}{2}x^T(0)Px(0) \quad (7.236)$$

Comparing equation(7.236) with equation(7.177), they have same form, but their solution is not different.

Feedback matrix K can not be directly solved from Lyapunov function in equation(7.237) because matrices P and K are both unknown.

$$(A-BKC)P+P(A-BKC)=-Q-C^T K^T RKC \quad (7.237)$$

A simple solving method is gradient reducing method. Matrix P may be expressed by matrix K from equation(7.237). Then matrix expression $P(K)$ is replaced into equation(7.236). And then order

$$\frac{\partial J(K)}{\partial K}=0 \quad (7.238)$$

Then matrix K will be solved from equation(7.238). Here, what needs to be noted is that the optimal parameters of feedback matrix K are related to original state condition $x(0)$.

Example7.18 A given state equation of linear time-constant continuous system is as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

System output equation:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Performance index functional J is

$$J = \frac{1}{2}\int_0^\infty \left[x^T Q x + u^T R u\right] dt, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Assume that system state $x_2(t)$ is not measured, and then output feedback should be adopted here. Please find feedback matrix k to make that functional J gets minimum.

Solution:

From equation(7.228), state equation of closed-loop system can be gained:

$$\dot{x} = \bar{A}x = [A - BKC]x = \left\{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} k \begin{bmatrix} 1 & 0 \end{bmatrix}\right\} x = \begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix} x$$

The performance index functional J from equation(7.229) can be written out:

$$J = \frac{1}{2}\int_0^\infty x^T(t)\bar{Q}x(t)dt = \frac{1}{2}\int_0^\infty x^T(t)\left[Q + C^T K^T R K C\right]x(t)dt$$

$$= \frac{1}{2}\int_0^\infty x^T(t)\begin{bmatrix} 1+k^2 & 0 \\ 0 & 1 \end{bmatrix} x(t)dt$$

According to Lyapunov equation(7.231), we can write out the following equality:

$$\bar{A}^T P + P\bar{A} = -\bar{Q}$$

Namely,

$$\begin{bmatrix} 0 & -k \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix} = -\begin{bmatrix} 1+k^2 & 0 \\ 0 & 1 \end{bmatrix}$$

Merge and then gain

$$\begin{bmatrix} -2kp_{12} & p_{11} - p_{12} - kp_{22} \\ p_{11} - p_{12} - kp_{22} & 2(p_{12} - p_{22}) \end{bmatrix} = -\begin{bmatrix} 1+k^2 & 0 \\ 0 & 1 \end{bmatrix}$$

The following algebraic equation group can be gained from above equality:

$$\begin{cases} 2kp_{12} = 1 + k^2 \\ p_{11} - p_{12} - kp_{22} = 0 \\ 2(p_{12} - p_{22}) = -1 \end{cases}$$

All elements in matrix P can be expressed by feedback gain matrix k as follows:

$$p_{11} = \frac{1 + k + 2k^2 + k^3}{2k} \qquad p_{12} = \frac{1+k^2}{2k}, \qquad p_{22} = \frac{1+k^2+k}{2k}$$

Namely,

$$P=\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}=\begin{bmatrix} \dfrac{1+k+2k^2+k^3}{2k} & \dfrac{1+k^2}{2k} \\ \dfrac{1+k^2}{2k} & \dfrac{1+k+k^2}{2k} \end{bmatrix}$$

Assume original value of system state: $x_1(0)=1$, $x_2(0)=0$. And then the following expression from equation(7.236) can be written out:

$$J=\frac{1}{2}x^T(0)Px(0)=\frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} \dfrac{1+k+2k^2+k^3}{2k} & \dfrac{1+k^2}{2k} \\ \dfrac{1+k^2}{2k} & \dfrac{1+k+k^2}{2k} \end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}=\frac{1+k+2k^2+k^3}{4k}$$

Order

$$\frac{\partial J(K)}{\partial K}=\frac{1}{4}\frac{d}{dk}\left[\frac{1+k+2k^2+k^3}{k}\right]=\frac{2k^3+2k^2-1}{4k^2}=0$$

Hence, $k=0.5652$

7.9 Optimal Control of Discrete Control System

7.9.1 Basic form of linear discrete system

Consider the following discrete state equation:

$$\begin{aligned} &x(k+1)=f[x(k),\quad u(k),\quad k],\quad k=0,1,2,\cdots,N-1 \\ &x(t_0)=x_0 \end{aligned} \tag{7.239}$$

Here, $x(k)$ is the system state of n dimension state vector at kth time; $u(k)$ is system control input of r dimension control vector between kth step and $(k+1)th$ step.

Optimal control problem is to ensure optimal control sequence $\{u(0),\quad u(1),\quad \cdots,\quad u(N-1)\}$ to make the following performance index functional minimum.

$$J=\phi[x(N)]+\sum_{k=0}^{N-1}L[x(k),\quad u(k),\quad k] \tag{7.240}$$

Here, discrete system state $x(N)$ is assumed to be free. $\phi[x(N)]$ shows the consuming cost in aim function.

Such problem is the same as previous control problem in nature. Their difference is that the number of variable increases N times. Namely,

$$\{x_1(k),\quad x_2(k),\quad \cdots,\quad x_n(k)\},\quad (k=1,2,\cdots,N)$$

$$\{u_1(k),\quad u_2(k),\quad \cdots,\quad u_n(k)\},\quad (k=0,1,2,\cdots,N-1)$$

Constraining equation(7.239) adds N times, too; hence problem scale also expands.

Imitating previous idea, expression(7.239) may be written into the following form:

$$f[x(k),\quad u(k),\quad k]-x(k+1)=0 \tag{7.241}$$

Number of Lagrange undetermined constants adds N times, too.

$$\{\lambda_1(k),\quad \lambda_2(k),\quad \cdots,\quad \lambda_n(k)\},\quad (k=1,2,\cdots,N)$$

Imitating previous methods, we now construct a new function:

$$V=\phi[x(N)]+\sum_{k=0}^{N-1}\{L[x(k),\quad u(k),\quad k]+\lambda^T(k+1)[f[x(k),\quad u(k),\quad k]-x(k+1)]\} \tag{7.242}$$

Then above minimum problem of equation(7.240) is transformed into finding the minimum problem of no constrained problem.

In order to conveniently remark relative expressions, we give out the following marks:

$$L_k[x(k),\quad u(k)]=L[x(k),\quad u(k),\quad k] \tag{7.243}$$

$$f_k[x(k),\quad u(k)]=f[x(k),\quad u(k),\quad k] \tag{7.244}$$

$$H_k=L_k[x(k),\quad u(k)]+\lambda^T(k+1)f_k[x(k),\quad u(k)] \tag{7.245}$$

Then equation(7.242) may be simplified into the following form:

$$V=\phi[x(N)]-\lambda^T(N)x(N)+H_0+\sum_{k=1}^{N-1}\left[H_k-\lambda^T(k)x(k)\right] \tag{7.246}$$

After we find the increment of functional V, the following linear master unit can be gained by neglecting high-order infinitesimal term.

$$\begin{aligned}\Delta V&=\Delta x^T(N)\left[\frac{\partial\phi_N}{\partial x(N)}-\lambda(N)\right]+\Delta x^T(0)\left[\frac{\partial H_0}{\partial x(0)}\right]+\Delta u^T(0)\left[\frac{\partial H_0}{\partial u(0)}\right]\\&+\sum_{k=1}^{N-1}\Delta x^T(k)\left[\frac{\partial H_k}{\partial x(k)}-\lambda(k)\right]+\sum_{k=1}^{N-1}\Delta u^T(k)\left[\frac{\partial H_k}{\partial u(k)}\right]\end{aligned} \tag{7.247}$$

Necessary condition that functional V gains minimum is $\Delta V=0$. After we consider that expression $x(0)=x_0$ is constant, there is expression $\Delta x(0)=0$. Then minimum necessary condition can be achieved from equation(7.247) as follows:

$$\begin{cases}\dfrac{\partial H_k}{\partial x(k)}=\lambda(k),\quad (k=1,2,\cdots,N-1)\\ \dfrac{\partial H_k}{\partial u(k)}=0,\quad (k=1,2,\cdots,N-1)\\ x(k+1)=f[x(k),\quad u(k)],\quad (k=1,2,\cdots,N-1)\\ x(0)=x_0\\ \dfrac{\partial\phi[x(N)]}{\partial x(N)}=\lambda(N)\end{cases} \tag{7.248}$$

Namely,

$$\begin{cases} \dfrac{\partial L[x(k),\ u(k),\ k]}{\partial x(k)} + \left[\dfrac{\partial f[x(k),\ u(k),\ k]}{\partial x(k)}\right]^T \lambda(k+1) = \lambda(k), \quad (k=1,2,\cdots,N-1) \\ \dfrac{\partial L[x(k),\ u(k),\ k]}{\partial u(k)} + \left[\dfrac{\partial f[x(k),\ u(k),\ k]}{\partial u(k)}\right]^T \lambda(k+1) = 0, \quad (k=1,2,\cdots,N-1) \\ x(k+1) = f[x(k),\ u(k)], \quad (k=0,1,2,\cdots,N-1) \\ x(0) = x_0 \\ \dfrac{\partial \phi[x(N)]}{\partial x(N)} = \lambda(N) \end{cases} \tag{7.249}$$

The number of equation(7.48) or (7.49) are respectively $n(N-1)$, rN, nN, n and n. Total number of equation is $(2n+r)N+n$. Except n variables $x(0)$, the total undetermined variable number is $(2n+r)N$.

In equation(7.248) or (7.249), given $\lambda(N)$ at final time N and $x(0)$ at initial time, are called to their boundary condition. The problem of solving the boundary of given two points is called to the problem of two point boundary value.

7.9.2 State regulator of linear discrete system

Assume that the state equation of linear discrete system is the following:

$$\begin{aligned} &x(k+1) = A(k)x(k) + B(k)u(k), \quad (k=0,1,2,\cdots,N-1) \\ &x(t_0) = x_0 \end{aligned} \tag{7.250}$$

In this section, we will respectively discuss two cases: finite time state regulator and infinite time state regulator.

1.Finite time state regulator

For finite time state regulator, quadratic performance index functional J can be depicted as follows

$$J = \frac{1}{2}x^T(N)Px(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left[x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\right] \tag{7.251}$$

Here, control sequence $u(k)$ is not constrained; weighting matrices P and Q are positive semidefinite matrices; matrix R is positive definite symmetric matrix.

Please find the optimal control sequence $u^*(k)(k=0,1,2,\cdots,N-1)$ to make performance index functional J minimum.

Above optimal control problem may be solved by minimum principle. And now let's firstly construct Hamilton function:

$$H(x,\ u,\ \lambda,\ k) = \frac{1}{2}x^T(k)Q(k)x(k) + \frac{1}{2}u^T(k)R(k)u(k) + \lambda^T(k+1)[A(k)x(k) + B(k)u(k)]$$

$$k = 0,1,2,\cdots,N-1 \tag{7.252}$$

Then the following canonical equation can be achieved:

$$x(k+1)=\frac{\partial H(k)}{\partial \lambda(k+1)}=A(k)x(k)+B(k)u(k) \tag{7.253}$$

$$\lambda(k)=-\frac{\partial H(k)}{\partial x(k)}=Q(k)x(k)+A^T(k)\lambda(k+1) \tag{7.254}$$

Final performance index is

$$\theta[x(N),\quad N]=\frac{1}{2}x^T(N)Px(N) \tag{7.255}$$

Hence, boundary condition and transverse condition are as follows:

$$x(0)=x_0 \tag{7.256}$$

$$\lambda(N)=\frac{\partial}{\partial x(N)}\left[\frac{1}{2}x^T(N)Px(N)\right]=Px(N) \tag{7.257}$$

If control law $u(t)$ is not constrained, the following control equation is not satisfied:

$$\frac{\partial H(k)}{\partial N(k)}=R(k)u(k)+B^T(k)\lambda(k+1)=0 \tag{7.258}$$

Hence, we can get:

$$u^*(k)=-R^{-1}(k)B^T(k)\lambda(k+1) \tag{7.259}$$

In equation(7.257), there is linear relationship between $\lambda(N)$ and $x(N)$; and furthermore canonical equation is also linear, hence we can postulate that $x(k)$ and $\lambda(k)$ exist the following linear relation for any time:

$$\lambda(k)=K(k)x(k);\quad k=0,1,2,\cdots,N-1 \tag{7.260}$$

Here, matrix $K(k)$ is $n\times n$ undetermined matrix.

Equation(7.259) and equation(7.260) are replaced into canonical equations(7.114) and (7.115); and then we can gain:

$$\begin{aligned}x(k+1)&=A(k)x(k)-B(k)R^{-1}(k)B^T(k)\lambda(k+1)\\&=A(k)x(k)-B(k)R^{-1}(k)B^T(k)K(k+1)x(k+1)\end{aligned} \tag{7.261}$$

$$\begin{aligned}\lambda(k)&=K(k)x(k)=Q(k)x(k)+A^T(k)\lambda(k+1)\\&=Q(k)x(k)+A^T(k)K(k+1)x(k+1)\end{aligned} \tag{7.262}$$

$x(k+1)$ is eliminated from equation(7.261) and equation(7.262); and relative result is compared with equation(7.260). Then we may get the following equation:

$$\begin{aligned}K(k)x(k)&=Q(k)x(k)+A^T(k)\lambda(k+1)\\&=Q(k)x(k)+A^T(k)K(k+1)x(k+1)\\&=Q(k)x(k)+A^T(k)K(k+1)\left[E+B(k)R^{-1}(k)K(k+1)\right]^{-1}A(k)x(k)\end{aligned} \tag{7.263}$$

Equation(7.263) should be found for any system state $x(k)$, hence $K(k)$ should satisfy the following equation:

$$K(k)=Q(k)+A^T(k)K(k+1)\left[E+B(k)R^{-1}(k)K(k+1)\right]^{-1}A(k) \tag{7.264}$$

Namely,

$$K(k)=Q(k)+A^{T}(k)\left[K^{-1}(k+1)+B(k)R^{-1}(k)B^{T}(k)\right]^{-1}A(k) \tag{7.265}$$

And furthermore the following final condition has to be satisfied:

$$K(N)=P \tag{7.266}$$

Equation(7.265) is rendered to Riccati difference equation. Optimal gain matrix sequence $K(k)$ may be confirmed through solving Riccati difference equation according to inverse time direction. After $K(k)$ is ensured, equation(7.262) can be utilized to gain

$$\lambda(k+1)=A^{-T}(k)[K(k)-Q(k)]x(k) \tag{7.267}$$

Hence, optimal control law is as follows:

$$u^{*}(k)=-R^{-1}(k)B^{T}(k)A^{-T}(k)[K(k)-Q(k)]x(k),\quad k=0,1,\cdots,N-1 \tag{7.268}$$

Equation(7.268) is replaced into equation

$$\begin{aligned}&x(k+1)=\{A(k)-B(k)R^{-1}(k)B^{T}(k)A^{-T}(k)[K(k)-Q(k)]\}x(k),\quad (k=0,1,2,\cdots,N-1)\\&x(t_0)=x_0\end{aligned} \tag{7.269}$$

Hence, optimal state $x^{*}(t)$ is the solution of linear difference equation(7.269). Optimal control $u^{*}(k)$ is unique, which is determined by equation(7.268). Optimal control $u^{*}(k)$ and optimal state $x^{*}(k)$ are substituted into performance index function, and we can acquire the minimum of performance index function:

$$J^{*}=\frac{1}{2}x^{T}(0)K(0)x(0) \tag{7.270}$$

State regulator block diagram of discrete system is shown in figure7.3.

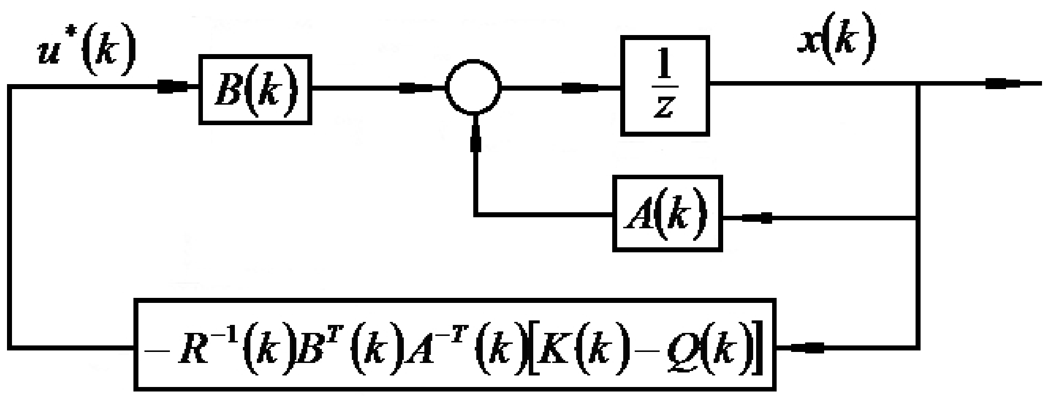

Figure7.3 State regulator block diagram of discrete system

2.Infinite time state regulator

For infinite-time state regulator $(N\to\infty)$, performance index functional is as follows:

$$J=\frac{1}{2}\sum_{k=0}^{\infty}\left[x^{T}(k)Qx(k)+u^{T}(k)Ru(k)\right] \tag{7.271}$$

Final time is possibly tendency to infinity, hence we have to require that the final state $x(N)$ should tend toward zero state. Otherwise, performance index functional does not converge. So that performance index functional does not include term $x(N)$.

When $N\to\infty$, Riccati matrix $K(k)$ will become a constant matrix, namely:

$$\lim_{k\to\infty} K(k) = K \tag{7.272}$$

Hence Riccati equation may be expressed into the following Riccati equation:

$$K = A^T KA + Q - A^T KB\left(R + B^T KB\right)^{-1} B^T KA \tag{7.273}$$

Optimal control law is

$$u^*(k) = -R^{-1}B^T A^{-T}(K - Q)x(k) \tag{7.274}$$

Optimal performance index functional yields to

$$J_\infty^* = \frac{1}{2}x^T(0)Kx(0) \tag{7.275}$$

Example7.19 A given state equation of linear discrete system is as follows

$$x(k+1) = x(k) + u(k); \quad k = 0,1,2,\cdots,N-1$$

Initial condition is $x(0)$, control law $u(k)$ is not constrained.

Performance index functional J is

$$J = \frac{1}{2}cx^2(N) + \frac{1}{2}\sum_{k=0}^{N-1} u^2(N)$$

Please find optimal control sequence $u^*(k)(k = 0,1,\cdots,N-1)$ to make that performance index functional J gets minimum.

Solution:

In order to demonstrate given conclusions in this section, we assume $N = 2$, namely only solve a two-step problem. Hence its performance index functional is the following:

$$J = \frac{1}{2}cx^2(2) + \frac{1}{2}u^2(0) + \frac{1}{2}u^2(1)$$

Comparing with equation(7.250) and equation(7.251), we can know that the optimal control parameters are as follows:

$$A(k) = 1, \quad B(k) = 1, \quad P = c, \quad Q = 0 \text{ and } R = 1$$

From equation(7.265), Riccati equation is the following:

$$K(k) = \left[K^{-1}(k+1) + 1\right]^{-1} = \frac{K(k+1)}{K(k+1)+1}$$

Since $N = 2$ and $K(2) = P = c$; we can solve $K(k)(k = 0,1)$ in terms of inverse time:

$$K(1) = \frac{K(2)}{K(2)+1} = \frac{c}{c+1}, \qquad K(0) = \frac{K(1)}{K(1)+1} = \frac{c}{2c+1}$$

The optimal control can be gained from equation(7.268):

$$u^*(k) = -K(k)x(k); \quad k = 0,1$$

$$u^*(0) = -K(0)x(0) = -\frac{c}{2c+1}x(0)$$

Optimal state under the optimal control $u^*(0)$ is the following:

$$x^*(1) = x(0) + u^*(0) = \frac{c+1}{2c+1} x(0)$$

$$u^*(1) = -K(1)x^*(1) = -\frac{c}{c+1} x^*(1) = -\frac{c}{2c+1} x(0)$$

Optimal state under the optimal control $u^*(1)$ is the following:

$$x^*(2) = x^*(1) + u^*(1) = \frac{1}{2c+1} x(0)$$

Optimal performance index functional is

$$J^* = \frac{1}{2} K(0) x^2(0) = \frac{c}{2(2c+1)} x^2(0)$$

7.9.3 Optimal Control Discretization of linear continuous system

Assume that the state equation of continuous system is the following:

$$\dot{x} = f[x(t), \quad u(t), \quad t] \qquad x(t_0) = x_0 \tag{7.276}$$

Aim functional is the following:

$$J = \theta[x(t_f), \quad t_f] + \int_{t_0}^{t_f} L[x(t), \quad u(t), \quad t] dt \tag{7.277}$$

Here, $\theta[x(t_f)]$ is final function; final state $x(t_f)$ is free.

Optimal control problem is to find optimal control law $u^*(t)$ to make equation(7.277) minimum under the condition of equation(7.276). We should firstly disperse equation (7.276) and then find the optimal solution of discrete system.

Firstly, equation(7.276) is transformed into the following discrete form:

$$x(k+1) - x(k) = f[x(k), \quad u(k), \quad k]\Delta t; \quad (k = 0,1,\cdots,N-1) \tag{7.278}$$

Performance index functional J is the following:

$$J = \sum_{k=0}^{N-1} L[x(k), \quad u(k), \quad k]\Delta t + \theta[x(N)] \tag{7.279}$$

Here, Δt is system sampling time.

Then, find optimal solution of discrete system. If we make that system sampling time Δt tends toward zero, discrete system will become a continuous system.

Similarly, we may imitating previously processing method, and then give out the following definition of Hamilton function:

$$H[x(k), \quad u(k), \quad \lambda(k), \quad k] = L[x(k), \quad u(k), \quad k] + \lambda^T(k+1) f[x(k), \quad u(k), \quad k] \tag{7.280}$$

The necessary condition of getting minimum is the following:

$$\left.\begin{array}{c}\frac{\partial H}{\partial x(k)}\Delta t=-\lambda(k+1)+\lambda(k)\\ \frac{\partial H}{\partial u(k)}\Delta t=0\\ f[x(k),\quad u(k),\quad k]\Delta t=x(k+1)-x(k)\\ x(t_0)=x_0\\ \frac{\partial\theta[x(N)]}{\partial x(N)}=\lambda(N)\end{array}\right\} \qquad (7.281)$$

When $\Delta t \to 0$, equation(7.281) will become the following form:

$$\left.\begin{array}{c}\frac{\partial H}{\partial x(t)}=-\lambda(t)\\ \frac{\partial H}{\partial u(t)}=0\\ \dot{x}=f[x(t),\quad u(t),\quad t]\quad or\ \frac{\partial H(t)}{\partial\lambda(t)}=\dot{x}(t)\\ x(t_0)=x_0\\ \frac{\partial\theta[x(t_f)]}{\partial x(t_f)}=\lambda(t_f)\end{array}\right\} \qquad (7.282)$$

Here, $H(t)=L[x(t),\quad u(t),\quad t]+\lambda^T(t)f[x(t),\quad u(t),\quad t]$.

Equation(7.282) is the necessary condition of optimal solution in equation(7.276) and equation(7.277).

7.10 Dynamic Programming

In 1950s, dynamic programming was brought forward by America Scholar R.Bellman for solving optimal control problem of multistage decision process.Dynamic programming has been widely applied into plenty of technical fields and control engineering.

7.10.1 Optimality principle

In nature, dynamic programming is to solve the optimization of multistage decision process on the basis of optimality principle. Here, multistage decision process means that a whole course is divided into plenty of stages and that in each stage the decision needs to be given to make whole course realize optimization. That's to say, an optimal policy has the property that whatever initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state result from the first decision. As used here, a "decision" is a choice of control at a particular time and the "policy" is the entire control sequence or function.

Consider the nodes in figure7.4, as states, in general sense. A decision is the choice of alternative paths leaving a given node. The goal is to move from A node to B node with minimum cost.

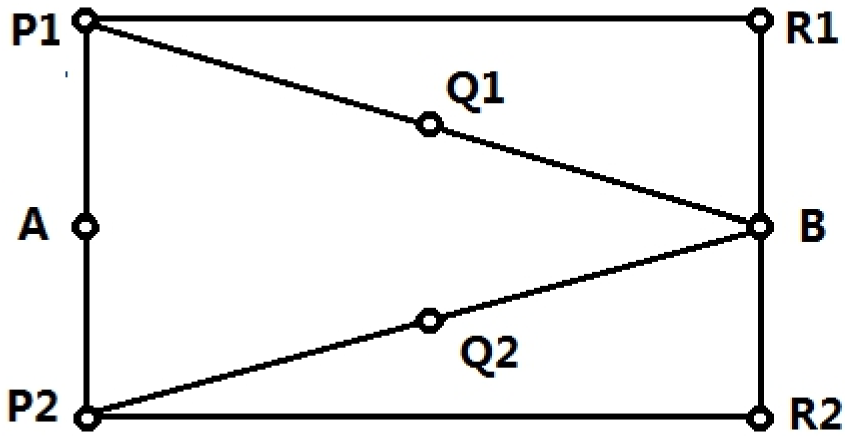

Figure7.4 Route diagram from A node to B node

A cost is associated with each segment of the line graph. Define J_{ab} as the cost between node A and B node. J_{AP1} is the cost between node A and node $P1$, etc. For path $A \to P1 \to R1 \to B$, the total cost is $J = J_{ab} = J_{AP1} + J_{P1R1} + J_{R1B1}$. Optimal path(policy) is defined by the following expression.

$$\min J = \min\left[J_{AP1} + J_{P1R1} + J_{R1B},\quad J_{AP1} + J_{P1Q1} + J_{Q1B},\quad J_{AP2} + J_{P2R2} + J_{R2B},\quad J_{AP2} + J_{P2Q2} + J_{Q2B}\right] \tag{7.282}$$

If initial state is node A and if initial decision is to go to node $P1$, then the path from node $P1$ to node B must certainly be selected optimally if the overall path from node A to node B is to be optimum. If final decision is to go to node $P2$, the path from node $P2$ to node B must then be selected optimally.

Let's postulate g_{P1} and g_{P2} be the minimum costs from node $P1$ and cost $P2$, respectively, to node B. Then $g_{P1} = \min\left[J_{P1Q1} + J_{Q1B},\quad J_{P1R1} + J_{R1B}\right]$ and $g_{P2} = \min\left[J_{P2Q2} + J_{Q2B},\quad J_{P2R2} + J_{R2B}\right]$. The principle of optimality allows equation(7.282) be written as

$$g_a = \min\left[J_{AP1} + g_{P1},\quad J_{AP2} + g_{P2}\right] \tag{7.283}$$

In equation(7.283), the key feature is that the quantity be minimized consists of two parts:

(1) The part directly attributable to the current decision, such as J_{AP1} and J_{AP2}.

(2) The part representing the minimum value of all future costs, starting with the state which results from the first decision.

The principle of optimality replaces a choice between all alternatives (Eq.(7.282)) by a sequence of decisions between fewer alternatives(find g_{P1}, g_{P2} and then g_a from Equation(7.283)). Dynamic programming allows us to concentrate on a sequence of current decisions rather than being concerned about all decisions simultaneously.

Division of cost into the two parts, current and future, is typically, but these parts do not necessarily appears as a sum. A simple example illustrates the sequence nature of the method.

Example7.20 Given N numbers x_1, x_2, $\cdots$, x_N, find the smallest one.

Rather than consider all N numbers simultaneously, define g_k as the minimum of number x_k through x_N. Then $g_N = x_N$, Continuing to choose between two alternatives eventually leads to $g_1 = \min\{x_1,\ g_2\} = \min\{x_1,\ x_2,\ \cdots,\ x_N\}$.

The desired result g_1 need not be unique, since more than one number may have the same smallest value. The recursive nature of the formula $g_k = \min\{x_k,\ g_{k-1}\}$ is typically of all discrete dynamic programming solutions.

From above example, optimality principle can be summarized as follows:

Optimal strategies of a multistage decision course have such property whatever initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state result from the first decision.Optimal problem of solving N-grade decision course based on optimality principle is in nature to be simplified to solve the optimal problem of solving N single-grade decision courses. In solving optimal problem, positive calculation method can be adopted, inverting calculation method may be also utilized. But for time-variant system, inverting calculation method has to be adopted.

The steps of applying dynamic programming to solve optimal problem is as follows:

Step1 Optimal problem which needs to be solved must be transformed into a multistage decision course.

Step2 On the basis of optimality principle, the function equation of multistage decision course is confirmed, which should include relative constraint conditions.

Step3 Recursive solution of function equation is solved.

7.10.2 Discrete optimality principle

1. Discretely optimal problem

We postulate that state difference equation of discrete system is as follows:

$$x(k+1) = f[x(k),\ u(k),\ k],\quad k = 0,1,2,\cdots,N-1 \tag{7.284}$$

Performance index functional in each step transformation is expressed as:

$$J = J[x(k),\ u(k)] \tag{7.285}$$

Initial state is $x(0)$ and state vector $x(k)$ at kth time is $n\times 1$ matrix; $u(k)$ is permissible control vector at kth time, which is $r\times 1$ matrix and which can be constrained or not.

Applying dynamic programming method to solve discretely optimal control problem.First-level decision course is to confirm that first-level state is transformed from initial state $x(0)$ under the control law $u(0)$:

$$x(1) = f[x(0), \quad u(0)] \tag{7.286}$$

Performance index functional:

$$J_1 = J[x(0), \quad u(0)] \tag{7.287}$$

Optimal control $u^0(0)$ is confirmed in minimum. When the first-level state is converted into the second-level state:

$$x(2) = f[x(1), \quad u(1)] = f[f[x(0), \quad u(0)], \quad u(1)] \tag{7.288}$$

Performance index function is the following:

$$J[x(1), \quad u(1)] = J[f[x(0), \quad u(0)], \quad u(1)] \tag{7.289}$$

General performance index functional of above two step transformation is as follows:

$$J_2 = J[x(0), \quad u(0)] + J[f[x(0), \quad u(0)], \quad u(1)] \tag{7.290}$$

From equation(7.290), it's easily observed that performance index functional J_2 is only the function of control vector sequence $u(0)$ and $u(1)$ for initial state $x(0)$ is given. Then selecting control vector sequence $u(0)$ and $u(1)$ is required to make performance index functional J_2 get minimum. This is two-level decision course.

Similarly, for N -level decision course, the state vector sequence of system is as follows:

$$\begin{aligned} x(1) &= f[x(0), \quad u(0)] \\ x(2) &= f[x(1), \quad u(1)] = f[f[x(0), \quad u(0)], \quad u(1)] \\ x(3) &= f[x(2), \quad u(2)] = f[f[f[x(0), \quad u(0)], \quad u(1)], \quad u(2)] \\ &\cdots\cdots\cdots\cdots \\ x(N) &= f[x(N-1), \quad u(N-1)] = f[x(0), \quad u(0), \quad u(1), \quad \cdots \quad u(N-2) \quad u(N-1)] \end{aligned} \tag{7.291}$$

Relative performance index functional is the following:

$$\begin{aligned} J_1 &= J[x(0), \quad u(0)] \\ J_2 &= J[x(0), \quad u(0)] + J[x(1), \quad u(1)] \\ &\cdots\cdots\cdots \\ J_N &= J[x(0), \quad u(0)] + J[x(1), \quad u(1)] + \cdots + J[x(N-1), \quad u(N-1)] = \sum_{k=0}^{N-1} J[x(k), \quad u(k)] \end{aligned} \tag{7.292}$$

Equation(7.292) demonstrates that when initial state $x(0)$ is given, performance index functional J_N is only the function of control vector sequence $u(k)(k = 0,1,2\cdots\cdots N-1)$, and that system state $x(N)$ is decided by equation(7.291). N -level decision course is to select control vector sequence $\{u(0), \quad u(1), \quad \cdots, \quad u(N-1)\}$ make performance index functional in equation(7.292) get minimum.

On the basis of optimality principle, the optimization of N -level decision course requires that however the first-level control vector $u(0)$ is confirmed, remaining $(N-1)-level$ course must constitute $(N-1)-level$ optimal course from the beginning of system state $x(1) = f[x(0), \quad u(0)]$ under the control law $u(0)$. If sign $J_N^0[x(0)]$ is marked as the minimum of performance index functional J_N in $N-level$ decision course, sign $J_{N-1}^0[x(1)]$ is the minimum of performance index functional J_{N-1} in $(N-1)-level$ decision course; then the following recursive function equation can be gained, namely

$$J_N^0[x(0)] = \min\{J[x(0), \quad u(0)] + J_{N-1}^0[f[x(0), \quad u(0)]]\} \tag{7.293}$$

Recursive equation may be gained from equation(7.293) to acquire optimal control strategy or optimal control sequence $\{u^0(0), \quad u^0(1), \quad \cdots \quad u^0(N-1)\}$. Equation(7.293) is called to Bellman functional equation or dynamic programming basic equation.

Generally, function equation(7.293) is very complicate and it needs to be solved to acquire recursive solution by digital computer.

Example7.21 State difference equation of discrete system is the following:

$$x(k+1) = 2x(k) + u(k), \quad x(0) = 1$$

Please try to ensure optimal control sequence $u(0), \quad u(1), \quad u(2)$ to make the following functional get minimum.

$$J = \sum_{k=0}^{2}\left[x^2(k) + u^2(k)\right]$$

Solution:

In this example, we can apply dynamic programming to solve the optimal control sequence $u^0(0), \quad u^0(1), \quad u^0(2)$. If solving optimal control sequence is converted into solving the optimal problem of multistage decision course, the given discrete system can be considered as three-level decision course, namely $N = 3$.

Step1, solve the optimal control law $u^0(2)$ from the final level. Postulate that $x^0(2)$ is the final-level initial state to achieve relative input control $u^0(2)$. Then performance index functional J_2 is

$$J_2 = x^2(2) + u^2(2)$$

The derivative of above functional J_2 yields to the following form:

$$\frac{\partial J_2}{\partial u(2)} = \frac{\partial}{\partial u(2)}\left[x^2(2) + u^2(2)\right] = 2u(2) = 0$$

Final input control solution $u^0(2)$ can be achieved: $u^0(2) = 0$

Relative performance index functional $J_2^0[x(2)]$ is also gotten:

$$J_2^0[x(2)] = x^2(2)$$

Step2, inversely deduce it previous level. Then performance index functional J_1 is

$$J_1 = \left[x^2(1) + u^2(1)\right] + J_2 = \left[x^2(1) + u^2(1)\right] + \left[x^2(2) + u^2(2)\right]$$

In terms of optimality principle, have

$$J_1^0[x(1)] = \min_{u(1)}\{[x^2(1)+u^2(1)]+[x^2(2)+u^2(2)]\}$$
$$= \min_{u(1)}\{[x^2(1)+u^2(1)]+x^2(2)\}$$
$$= \min_{u(1)}\{[x^2(1)+u^2(1)]+[2x(1)+u(1)]^2\}$$
$$= \min_{u(1)}\{5x^2(1)+4x(1)u(1)+2u^2(1)\}$$

The derivative of performance index functional $J_1^0[x(1)]$ is the following:

$$\frac{\partial}{\partial u(1)}[5x^2(1)+4x(1)u(1)+2u^2(1)] = 4x(1)+4u(1) = 0$$

Its solution: $u_1^0(1) = -x(1)$

Performance index functional $J_1^0[x(1)]$: $J_1^0[x(1)] = 3x^2(1)$

Step3, inversely infer initial level. Then, performance index functional is:

$$J_0 = [x^2(0)+u^2(0)]+[x^2(1)+u^2(1)]+[x^2(2)+u^2(2)]$$

From optimality principle, we can know:

$$J_0^0[x(0)] = \min_{u(0)}\{[x^2(0)+u^2(0)]+J_1^0[x(1)]\}$$
$$= \min_{u(0)}\{x^2(0)+u^2(0)+3x^2(1)\}$$
$$= \min_{u(0)}\{x^2(0)+u^2(0)+3[2x(0)+u(0)]^2\}$$
$$= \min_{u(0)}\{13x^2(0)+12x(0)u(0)+4u^2(0)\}$$

The derivative of above performance index functional $J_0^0[x(0)]$ can be expressed as

$$\frac{\partial}{\partial u(0)}[13x^2(0)+12x(0)u(0)+4u^2(0)] = 12x(0)+8u(0) = 0$$

Solution: $u_0^0(0) = -\frac{3}{2}x(0)$

Functional: $J_0^0[x(0)] = 4x^2(0)$

Step4, get the optimal control strategy. From difference equation, the following equation can be gained:

$$x(1) = 2x(0)+u(0)$$

Hence, $u^0(1) = -x(1) = -2x(0)-u(0) = -\frac{1}{2}x(0)$

On consider $x(0)=1$, we can obtain the optimal control sequence of given discrete system.Namely, optimal control strategy is the following:

$$u^0(0) = -\frac{3}{2},\quad u^0(1) = -\frac{1}{2},\quad u^0(2) = 0$$

Example7.22 The transfer function of one-order inertia system is the following:

$$G(s) = \frac{K}{Ts+1}$$

Functional: $$J=\int_0^{t_f}\left(x^2+\gamma u^2\right)dt,\quad x(0)=x_0$$

Final system state $x(t_f)$ is free. Assume that discrete control is adopted here to divide control period $[0,\ t_f]$ into three segments.Please try to ensure optimal control sequence to make the given functional get minimum.

Solution:

Given transfer function is transformed into the following state equation:

$$\dot{x}=-\frac{1}{T}x+\frac{K}{T}u$$

Discretize above state equation, and then get the difference equation:

$$x(k+1)\Delta t=e^{-\frac{\Delta t}{T}}x(k)\Delta t+K\left(1-e^{-\frac{\Delta t}{T}}\right)u(k)\Delta t,$$ Δt is sampling period.

Order $g=e^{-\frac{\Delta t}{T}}$ and $h=K\left(1-e^{-\frac{\Delta t}{T}}\right)$; and then above expression yields to

$$x(k+1)=gx(k)+hu(k)$$

The discrete performance index functional is the following:

$$J=\sum_{k=0}^{2}\left[x^2(k)+\gamma u^2(k)\right]\Delta t$$

Apply dynamic programming to solve the optimal control problem for three given segments. Here, $J_3^0[x(3)]=0$, $\theta[x(3)]=0$.

Step1, deduce the optimal control law $u^0(2)$ from the final level. Then performance index functional is the following:

$$J_2=\left[x^2(2)+\gamma u^2(2)\right]\Delta t$$

$$\frac{\partial}{\partial u(2)}\left\{\left[x^2(2)+\gamma u^2(2)\right]\Delta t\right\}=2\gamma u(2)\Delta t=0$$

Optimal control $u^0(2)$: $u^0(2)=0$

Relative performance index functional $J_2^0[x(2)]$ is also gotten:

$$J_2^0[x(2)]=x^2(2)\Delta t$$

Step2, inversely deduce it previous level. Then performance index functional J_1 is

$$J_1=\left[x^2(1)+\gamma u^2(1)\right]\Delta t+J_2=\left\{\left[x^2(1)+\gamma u^2(1)\right]+\left[x^2(2)+\gamma u^2(2)\right]\right\}\Delta t$$

In terms of optimality principle, have

$$\begin{aligned}J_1^0[x(1)]&=\min_{u(1)}\left\{\left[x^2(1)+\gamma u^2(1)\right]+\left[x^2(2)+\gamma u^2(2)\right]\right\}\Delta t\\&=\min_{u(1)}\left\{\left[x^2(1)+\gamma u^2(1)\right]+x^2(2)\right\}\Delta t\\&=\min_{u(1)}\left\{\left[x^2(1)+\gamma u^2(1)\right]+\left[gx(1)+hu(1)\right]^2\right\}\Delta t\\&=\min_{u(1)}\left\{\left(1+g^2\right)x^2(1)+2ghx(1)u(1)+\left(\gamma+h^2\right)u^2(1)\right\}\Delta t\end{aligned}$$

The derivative of performance index functional $J_1^0[x(1)]$ is the following:

$$\frac{\partial}{\partial u(1)}\left[\left(1+g^2\right)x^2(1)+2ghx(1)u(1)+\left(\gamma+h^2\right)u^2(1)\right]\Delta t=\left[2ghx(1)+2\left(\gamma+h^2\right)u(1)\right]\Delta t=0$$

Its solution: $u_1^0(1)=-\dfrac{gh}{\gamma+h^2}x(1)$

Performance index functional $J_1^0[x(1)]$: $J_1^0[x(1)]=\left(1+\dfrac{rg^2}{r+h^2}\right)x^2(1)\Delta t$

Step3, inversely infer initial level. Then, performance index functional is:

$$J_0=\left[x^2(0)+\gamma u^2(0)\right]+\left[x^2(1)+\gamma u^2(1)\right]+\left[x^2(2)+\gamma u^2(2)\right]$$

From optimality principle, we can know:

$$J_0^0[x(0)]=\min_{u(0)}\left\{\left[x^2(0)+\gamma u^2(0)\right]+J_1^0[x(1)]\right\}\Delta t$$

$$=\min_{u(0)}\left\{x^2(0)+\gamma u^2(0)+\left(1+\frac{rg^2}{r+h^2}\right)x^2(1)\right\}\Delta t$$

$$=\min_{u(0)}\left\{x^2(0)+\gamma u^2(0)+\left(1+\frac{rg^2}{r+h^2}\right)[gx(0)+hu(0)]^2\right\}\Delta t$$

The derivative of above performance index functional $J_0^0[x(0)]$ can be expressed as

$$\frac{\partial J_0^0[x(0)]}{\partial u(0)}=\left\{2\gamma u(0)+2h\left(1+\frac{rg^2}{r+h^2}\right)[gx(0)+hu(0)]\right\}\Delta t=0$$

The solution of above differential equation can be solved:

$$u_0^0(0)=-\frac{gh\left(\gamma+h^2+\gamma g^2\right)}{\left(\gamma+h^2\right)^2+\gamma g^2h^2}x(0)$$

Step4, get the optimal control strategy. From difference equation, the following equation can be gained:

$$x(1)=gx(0)+hu(0)=\frac{gr\left(r+h^2\right)}{\left(r+h^2\right)^2+\gamma g^2h^2}x(0)$$

Hence, $u_1^0(1)=-\dfrac{gh}{\gamma+h^2}x(1)=-\dfrac{g^2hr}{\left(r+h^2\right)^2+\gamma g^2h^2}x(0)$

On consider $x(0)=x_0$, we can obtain the optimal control sequence of given discrete system.Namely, optimal input control strategy is the following:

$$u_0^0(0)=-\frac{gh\left(\gamma+h^2+\gamma g^2\right)}{\left(\gamma+h^2\right)^2+\gamma g^2h^2}x(0)=-\frac{gh\left(\gamma+h^2+\gamma g^2\right)}{\left(\gamma+h^2\right)^2+\gamma g^2h^2}x_0$$

$$u_1^0(1)=-\frac{g^2hr}{\left(r+h^2\right)^2+\gamma g^2h^2}x(0)=-\frac{g^2hr}{\left(r+h^2\right)^2+\gamma g^2h^2}x_0$$

Apparently, optimal input control sequences are the function of initial state variable.

2.Dynamic programming of discretely linear quadratic optimal problem

Assume that the state difference equation of linear time-invariant discrete system is the following:

$$x(k+1) = Ax(k) + Bu(k) \tag{7.294}$$

Please ensure the optimal control sequence $\{u_0^0(0),\ u_1^0(1),\ \cdots,\ u_{N-1}^0(N-1)\}$ to make the following performance index functional have minimum.

$$J = x^T(N)Px(N) + \sum_{k=0}^{N-1}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \tag{7.295}$$

Here, x is system state vector which is $n\times 1$ matrix; u is control vector which is $r\times 1$ matrix which is constrained or unconstrained permissible control; P and Q is weighting matrix which is positive semi-definite symmetric matrix; R is positive definite symmetric weighting matrix.

Firstly, we should consider the optimal control from final system state. The initial condition of final system state is $x(N-1)$, and then ensure optimal control law $u_{N-1}^0(N-1)$ to make the following performance index functional minimum:

$$J_{N-1}[x(N-1),\ u(N-1)] = x^T(N)Px(N) + x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1) \tag{7.296}$$

In order to conveniently discuss functional, mark

$$J_N^0[x(N)] = x^T(N)Px(N) = x^T(N)V_N x(N)$$

In terms of optimality principle, the minimum functional expression can be gained as

$$J_{N-1}[x(N-1),\ u(N-1)] = \min_{u(N-1)}\left\{x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1) + J_N^0[x(N)]\right\} \tag{7.297}$$

Equation(7.294) is replaced into equation(2.797) and then gain

$$\begin{aligned} J_{N-1}^0[x(N-1)] = &\min_{u(N-1)}\{x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1) \\ &+ [Ax(N-1) + Bu(N-1)]^T P[Ax(N-1) + Bu(N-1)]\} \end{aligned} \tag{7.298}$$

The terms of right side in equation(7.298) is partially differentiated from control function u.

$$\frac{\partial}{\partial u}\left(u^T Ru\right) = 2Ru\,, \qquad \frac{\partial}{\partial u}(Cu) = C^T\,, \qquad \frac{\partial}{\partial u}\left(u^T C\right) = C$$

Hence,

$$\frac{\partial J_{N-1}^0}{\partial u(N-1)} = 2Ru(N-1) + 2B^T PBu(N-1) + 2B^T PAx(N-1) = 0 \tag{7.299}$$

Then optimal control $u_{N-1}^0(N-1)$ which can make functional $J_{N-1}^0[x(N-1)]$ have minimum can be confirmed:

$$u_{N-1}^0(N-1) = -\left(R + B^T PB\right)^{-1} B^T PAx(N-1) \tag{7.300}$$

If the following expression is marked,

$$K_{N-1} = -\left(R + B^T PB\right)^{-1} B^T PA \tag{7.301}$$

Equation(7.301) can be written into the following form:

$$u_{N-1}^{0}(N-1)=-K_{N-1}x(N-1) \tag{7.302}$$

Equation(7.302) is replaced into equation(7.298) to get

$$\begin{aligned}J_{N-1}^{0}[x(N-1)]=\min_{u(N-1)}\{x^{T}(N-1)Qx(N-1)+x^{T}(N-1)K_{N-1}^{T}RK_{N-1}x(N-1)\\+x^{T}(N-1)[A-BK_{N-1}]^{T}V_{N}[A-BK_{N-1}]x(N-1)\}\end{aligned} \tag{7.303a}$$

Or equation(7.303a) is written into

$$J_{N-1}^{0}[x(N-1)]=x^{T}(N-1)V_{N-1}x(N-1) \tag{7.303b}$$

Here,

$$V_{N-1}=Q+K_{N-1}^{T}RK_{N-1}+[A-BK_{N-1}]^{T}V_{N}[A-BK_{N-1}] \tag{7.304}$$

Mathematical induction can be applied to be written for $l=1,2,3,\cdots,N$ from equation(7.301) to equation(7.304).

$$u_{l}^{0}(l)=-K_{l}x(l) \tag{7.305}$$

$$J_{l}^{0}[x(l)]=x^{T}(l)V_{l}x(l) \tag{7.306}$$

Among them, relative parameters can be expressed as

$$K_{l}=\left(R+B^{T}V_{l-1}B\right)^{-1}B^{T}V_{l-1}A \tag{7.307}$$

$$\begin{aligned}&V_{l}=Q+K_{l}^{T}RK_{l}+[A-BK_{l}]^{T}V_{l-1}[A-BK_{l}]\\&V_{N}=P\end{aligned} \tag{7.308}$$

Equation(7.307) is substituted into equation(7.308), and then achieve:

$$V_{l}=Q+A^{T}V_{l-1}A-A^{T}V_{l-1}B\left[R+B^{T}V_{l-1}B\right]^{-1}B^{T}V_{l-1}A \tag{7.309}$$

Or written into the following form:

$$V_{l}=Q+A^{T}M_{l}A \tag{7.310}$$

In equation(7.310),

$$M_{l}=V_{l-1}-V_{l-1}B\left[R+B^{T}V_{l-1}B\right]^{-1}B^{T}V_{l-1} \tag{7.311}$$

For $V_{N}=P$, matrix M_{N} can be confirmed from equation(7.310). From equation(7.311), matrix V_{N-1} can be calculated out. In terms of such recursive method, we can acquire matrices V_{l}, $\cdots$, V_{2} in turn. Then from equation(7.307), matrix K_{l} can be calculated out. Finally the optimal control sequence $u_{0}^{0}(0)$, $u_{1}^{0}(1)$, $\cdots$, $u_{N-1}^{0}(N-1)$ may be acquired from equation(7.305) in terms of given matrix K_{l}.

Summarizing above content, we can gain the following

Step1, order

$$V_{0}=P \tag{7.312}$$

Step2, for $l=1$, calculate the following matrices:

$$Z_{l}=Q+A^{T}V_{l-1}A \tag{7.313}$$

$$T_{l}=B^{T}V_{l-1}A \tag{7.315}$$

$$F_l = R + B^T V_{l-1} B \tag{7.316}$$

Step3, calculate

$$K_l = F_l^{-1} T_l \tag{7.317}$$

Step4, calculate the equation in terms of equation(7.309):

$$V_l = Z_l - T_l^T F_l^{-1} T_l \tag{7.318}$$

Step5, recalculate the content from step 2 to step4 for $l = N, N-1, \cdots, 3, 2$.

Step6, confirm the optimal control sequence

$$u_i^0(i) = -K_{N-i} x(i), \quad i = 0,1,2,\cdots,N-1 \tag{7.319}$$

From above calculation steps, we can easily know that dynamic programming can be applied to solve the linear quadratic optimal control problem of discrete system by state feedback. Here, matrix K_l is called to optimal feedback gain matrix which is only related to system matrix A, control matrix B and weighting matrices P, Q, R, not to initial condition.Hence, optimal feedback gain matrix K_l of linear quadratic optimal control problem in discrete system may be calculate offline.

Above steps are suitable for discrete time-variant system.

Example7.23 The state difference equation of discrete time-invariant system is as follows:

$$x(k+1) = x(k) + u(k), \quad x(0) = 10$$

Please try to ensure optimal control sequence to make the following functional get minimum.

$$J = x^2(2) + \sum_{k=0}^{1} \left[x^2(k) + u^2(k) \right]$$

Solution:

In terms of given information in this example, we can know:

$$A = 1, \quad B = 1, \quad P = 1, \quad Q = 1, \quad R = 1$$

$$J_2^0[x(2)] = x^T(2) P x(2) = x^T(2) V_0 x(2) = x^2(2), \quad V_0 = P = 1$$

According to the given steps from equation(7.313) to equation(7.319).

(1)When $l = 1$, calculate

$$Z_1 = Q + A^T V_0 A = 1 + 1 = 2$$

$$T_1 = B^T V_0 A = 1$$

$$F_1 = R + B^T V_0 B = 2$$

$$K_1 = F_1^{-1} T_1 = \frac{1}{2}$$

$$V_1 = Z_1 - T_1^T F_1^{-1} T_1 = 2 - \frac{1}{2} = 1.5$$

(2)When $l = 2$, calculate

$$Z_2 = Q + A^T V_1 A = 1 + 1.5 = 2.5$$

$$T_2 = B^T V_1 A = 1.5$$

$$F_2 = R + B^T V_1 B = 1 + 1.5 = 2.5$$

$$K_2 = F_2^{-1} T_2 = \frac{2}{5} \times \frac{3}{2} = 0.6$$

$$V_2 = Z_2 - T_2^T F_2^{-1} T_2 = \frac{5}{2} - \frac{3}{2} \times \frac{3}{5} = 1.6$$

(3) Give out the optimal control sequence $u_{N-i}^0(i), \quad i = 0,1$

$$u_0^0(0) = -K_{2-0} x(0) = -K_2 x(0) = -\frac{3}{5} \times 10 = -6$$

$$u_1^0(1) = -K_{2-1} x(1) = -K_1 x(1) = -K_1[x(0) + u(0)] = -\frac{1}{2} \times (10 - 6) = -2$$

7.10.3 Continuous optimality principle

Presume that the state equation of linear continuous system is as follows:

$$\dot{x} = f[x(t), \quad u(t), \quad t] \tag{7.320}$$

Initial state: $\quad x(t_0) = x_0$

The performance index functional J of linear continuous system is expressed as

$$J = \theta[x(t_f), \quad t_f] + \int_{t_0}^{t_f} L[x(t), \quad u(t), \quad t] dt \tag{7.321}$$

Please find the optimal control $u^0(t)$.

Under the condition of given final time t_f, if optimal control vector $U^0(t)$ has been confirmed, the minimum J^0 of performance index functional is only the scalar function of initial state $x(t_0)$ and initial time t_0; namely:

$$J^0[x(t), \quad t_f] = J[x(t_0), \quad t_0] = \theta[x^0(t_f), \quad t_f] + \int_{t_0}^{t_f} L[x^0(t), \quad u^0(t), \quad t] dt \tag{7.322}$$

In terms of optimality principle, if time t is a point in time period $[t_0, \quad t_f]$, the course from time t to final time t_f can be also divided into two-level course: $[t, \quad t+\Delta t]$ and $[t+\Delta t, \quad t_f]$, the minimum of performance index functional from time t to final time t_f can be written into the following form:

$$\begin{aligned} J^0[x, \quad t] &= \min_{u \in U} \left\{ \int_{t_0}^{t_f} L[x, \quad u, \quad t] dt + \theta[x(t_f)] \right\} \\ &= \min_{u \in U} \left\{ \int_{t_0}^{t+\Delta t} L[x, \quad u, \quad t] dt + \int_{t+\Delta t}^{t_f} L[x, \quad u, \quad t] dt + \theta[x(t_f)] \right\} \end{aligned} \tag{7.323}$$

If the course from time t to final time t_f is optimal, the rear sub-process from $t+\Delta t$

to final time t_f is also optimal, $t < t + \Delta t < t_f$. Hence, rear sub-process may be transformed into the following form:

$$J^0[x(t+\Delta t), \quad t+\Delta t] = \min_{u \in U}\left\{\int_{t+\Delta t}^{t_f} L[x, \quad u, \quad t]dt + \theta[x(t_f)]\right\} \tag{7.324}$$

When Δt is very small, have

$$\int_t^{t+\Delta t} L[x(t), \quad u(t), \quad t]dt = L[x(t), \quad u(t), \quad t]\Delta t \tag{7.325}$$

Then formula(7.323) can be approximately expressed into the following form:

$$J^0[x, \quad t] = \min_{u \in U}\{L[x(t), \quad u(t), \quad t]\Delta t + J^0[x(t+\Delta t) \quad t+\Delta t]\} \tag{7.326}$$

Expression $x(t+\Delta t)$ is developed into Taylor's polynomial form:

$$x(t+\Delta t) = x + \frac{dx}{dt}\Delta t + \frac{1}{2!}\frac{d^2x}{dt^2}(\Delta t)^2 + \cdots = x + \Delta x + \frac{1}{2!}(\Delta x)^2 \cdots$$

One-power polynomial is gotten from above polynomial, and then the following expressions can be acquired：

$$x(t+\Delta t) = x + \frac{dx}{dt}\Delta t = x + \Delta x$$

$$\Delta x = \frac{dx}{dt}\Delta t = f[x, \quad u, \quad t]\Delta t$$

$$J^0[x(t+\Delta t) \quad t+\Delta t] = J^0[x+\Delta x, \quad t+\Delta t]$$

At the adjacent field of function $[x, \quad t]$,above expression $J^0[x+\Delta x, \quad t+\Delta t]$ is expanded into Taylor series, too. expression $J^0[x+\Delta x, \quad t+\Delta t]$ is the x function and is related to time t. Hence, the corresponding Taylor's series expansion is the following:

$$J^0[x+\Delta x, \quad t+\Delta t] \approx J^0[x, \quad t] + \left[\frac{\partial J^0(x, \quad t)}{\partial x}\right]^T \Delta x + \frac{\partial J^0(x, \quad t)}{\partial t}\Delta t \tag{7.327}$$

Equation(7.327) is replaced into equation(7.326), and then achieve:

$$\begin{aligned} J^0[x, \quad t] &= \min_{u \in U}\left\{L[x, \quad u, \quad t]\Delta t + J^0[x, \quad t] + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T \Delta x + \frac{\partial J^0[x, \quad t]}{\partial t}\Delta t\right\} \\ &= J^0[x, \quad t] + \frac{\partial J^0[x, \quad t]}{\partial t}\Delta t + \min_{u \in U}\left\{L[x, \quad u, \quad t]\Delta t + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T f[x, \quad u, \quad t]\Delta t\right\} \end{aligned} \tag{7.328}$$

For $J^0[x, \quad t]$ is not related to input control law u, minimum of polynomial $J^0[x, \quad t]$ and $\frac{\partial J^0[x, \quad t]}{\partial t}$ may be placed into the outside of minimum polynomial. Move and merge, we can get:

$$-\frac{\partial J^0[x, \quad t]}{\partial t} = \min_{u \in U}\left\{L[x, \quad u, \quad t] + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T f[x, \quad u, \quad t]\right\} \tag{7.329}$$

Equation(7.329) is called to **Bellman equation or dynamic programming of**

continuous system. It is a partially differential equation on $J^0[x, \quad t]$. After solving equation(7.329), we can acquire the optimal control make functional J minimum. Its boundary condition is

$$J^0[x(t_f), \quad t_f] = \theta[x(t_f), \quad t_f] \tag{7.330}$$

If order

$$H(x, \quad u, \quad \lambda, \quad t) = L[x, \quad u, \quad t] + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T f[x, \quad u, \quad t]$$
$$= L[x, \quad u, \quad t] + \lambda^T f[x, \quad u, \quad t] \tag{7.331}$$

Here, $\lambda = \dfrac{\partial J^0(x, \quad t)}{\partial x}$ (7.332)

Then Equation(7.329) could be written into the following form:

$$-\frac{\partial J^0[x, \quad t]}{\partial t} = \min_{u \in U} H(x, \quad u, \quad \lambda, \quad t) \tag{7.333}$$

When control law $u(t)$ is not limited, then we may know:

$$-\frac{\partial J^0[x, \quad t]}{\partial t} = H(x, \quad u, \quad \lambda, \quad t) \tag{7.334}$$

Above equation is called to **Hamilton-Jacobi equation**. Above equation demonstrates that optimal control has to make Hamilton minimum. In fact, this is another form of minimum principle.

We may deduce transverse condition and adjoining condition from equation(7.329). Equation(7.329) can be written int the following form:

$$L[x, \quad u, \quad t] + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T f[x, \quad u, \quad t] + \frac{\partial J^0[x, \quad t]}{\partial t} = 0 \tag{7.335}$$

Partial differentiation is done for state variable x, and then we can get:

$$\frac{\partial L[x, \quad u, \quad t]}{\partial x} + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T \frac{\partial f[x, \quad u, \quad t]}{\partial x} + \left[\frac{\partial^2 J^0[x, \quad t]}{\partial^2 x}\right]^T f[x, \quad u, \quad t] + \frac{\partial^2 J^0[x, \quad t]}{\partial x \partial t} = 0 \tag{7.336}$$

The total derivative of expression $\dfrac{\partial J^0}{\partial x}$ for time t is the following:

$$\frac{d}{dt}\left[\frac{\partial J^0[x, \quad t]}{\partial x}\right] = \frac{\partial^2 J^0[x, \quad t]}{\partial x \partial t} + \frac{\partial^2 J^0[x, \quad t]}{\partial^2 x}\frac{dx}{dt} \tag{7.337}$$

Equation(7.337) is replaced into equation(7.336), and then we can gain

$$\frac{\partial L[x, \quad u, \quad t]}{\partial x} + \left[\frac{\partial J^0[x, \quad t]}{\partial x}\right]^T \frac{\partial f[x, \quad u, \quad t]}{\partial x} + \frac{d}{dt}\left[\frac{\partial J^0[x, \quad t]}{\partial x}\right] = 0 \tag{7.338}$$

If we order $\lambda(t) = \dfrac{\partial J^0[x, \quad t]}{\partial x}$, above equation may be written into the following form:

$$\frac{d\lambda(t)}{dt} = -\left\{\frac{\partial L[x,\quad u,\quad t]}{\partial x} + \left[\frac{\partial J^0[x,\quad t]}{\partial x}\right]^T \frac{\partial f[x,\quad u,\quad t]}{\partial x}\right\} \tag{7.339}$$

This is adjoining equation required $\dot{\lambda} = -\frac{\partial H}{\partial x}$, which is the same as the previous result.

When $t = t_f$, performance index functional at final time is the following:

$$J^0[x(t_f),\quad t_f] = \theta[x(t_f),\quad t_f] + \mu^T N[x(t_f),\quad t_f] \tag{7.340}$$

Here, μ and N have same dimension.

Partial differentiation is done for state variable $x(t)$, and then we can know:

$$\frac{\partial J^0[x(t_f),\quad t_f]}{\partial x(t_f)} = \left\{\frac{\partial \theta[x(t_f),\quad t_f]}{\partial x(t_f)} + \left[\frac{\partial N[x(t_f),\quad t_f]}{\partial x(t_f)}\right]^T \mu\right\}_{t=t_f} \tag{7.341}$$

Namely,

$$\lambda(t_f) = \left\{\frac{\partial \theta[x(t_f),\quad t_f]}{\partial x(t_f)} + \left[\frac{\partial N[x(t_f),\quad t_f]}{\partial x(t_f)}\right]^T \mu\right\}_{t=t_f} \tag{7.342}$$

Equation(7.340) is used to be partially differentiated for time variable t. And then we can get

$$\left.\frac{\partial J^0[x(t_f),\quad t_f]}{\partial t_f}\right|_{t=t_f} = \left\{\frac{\partial \theta[x(t_f),\quad t_f]}{\partial t_f} + \mu^T\left[\frac{\partial N[x(t_f),\quad t_f]}{\partial t_f}\right]\right\}_{t=t_f} \tag{7.343}$$

After considering equation(7.334) and equation(7.336), we can get

$$\left\{H + \frac{\partial \theta[x(t_f),\quad t_f]}{\partial t_f} + \mu^T\left[\frac{\partial N[x(t_f),\quad t_f]}{\partial t_f}\right]\right\}_{t=t_f} = 0 \tag{7.343}$$

Example7.24 Assume $\dot{x} = u$, please find optimal control $u^0(t)$ make the following performance index functional minimum.

$$\min_u J = \int_0^{t_f}\left(x^2 + \frac{1}{2}x^4 + u^2\right)dt$$

Solution:

In terms of given information, we can know:

$$L = x^2 + \frac{1}{2}x^4 + u^2, \qquad f(x,\quad u,\quad \lambda,\quad t) = u$$

Construct the following Hamilton function:

$$H = L + \frac{\partial J}{\partial x}f = x^2 + \frac{1}{2}x^4 + u^2 + \frac{\partial J}{\partial x}u$$

In terms of Hamilton-Jacobi equation in equation(7.333), we have the following equation:

$$-\frac{\partial J^0}{\partial t}=\min_{u\in u} H(x,\quad u,\quad \lambda,\quad t)=\min_{u\in u}\left(L+\frac{\partial J}{\partial x}f\right)=\min_{u\in u}\left(x^2+\frac{1}{2}x^4+u^2+\frac{\partial J}{\partial x}u\right)$$

After considering that control law u is not constrained, we can gain:

$$\frac{\partial H}{\partial u}=\frac{\partial\left(x^2+\frac{1}{2}x^4+u^2+\frac{\partial J^0}{\partial x}u\right)}{\partial u}=2u+\frac{\partial J^0}{\partial x}=0$$

Hence, $u^0=-\frac{1}{2}\frac{\partial J^0}{\partial x}$.

$$-\frac{\partial J^0}{\partial t}=\min_{u\in u}\left(x^2+\frac{1}{2}x^4+u^2+\frac{\partial J}{\partial x}u\right)=\min_{u\in u}\left(x^2+\frac{1}{2}x^4+\frac{1}{4}\left(\frac{\partial J}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2\right)$$

$$=\min_{u\in u}\left(x^2+\frac{1}{2}x^4-\frac{1}{4}\left(\frac{\partial J}{\partial x}\right)^2\right)$$

Boundary condition, $\theta[x(t_f)]=0$, hence, $J^0[x(t_f)]=0$.

If If we order $\lambda(t)=\frac{\partial J^0[x,\quad t]}{\partial x}$, we can know:

$$u^0=-\frac{1}{2}\lambda(t)$$

Example7.25 Assume the state equation of system is the following:

$$\dot{x}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

Initial state: $x(0)=\begin{bmatrix}1\\ 0\end{bmatrix}$

Performance index functional: $J=\int_0^{\infty}\left(2x_1^2+\frac{1}{2}u^2\right)dt$

please find optimal control $u^0(t)$ without constraint make the following performance index functional minimum.

Solution:

In terms of given information, we can know:

$$L=2x_1^2+\frac{1}{2}u^2,\qquad f=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

Construct the following Hamilton function:

$$H=L+\frac{\partial J}{\partial x}f=2x_1^2+\frac{1}{2}u^2+\begin{bmatrix}\frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2}\end{bmatrix}\begin{bmatrix}x_2\\ u\end{bmatrix}$$

$$=2x_1^2+\frac{1}{2}u^2+\frac{\partial J}{\partial x_1}x_2+\frac{\partial J}{\partial x_2}u$$

From Hamilton-Jacobi equation in equation(7.334), we can know:

$$-\frac{\partial J^0}{\partial t} = \min_{u\in u} H(x, \quad u, \quad \lambda, \quad t) = \min_{u\in u}\left(L + \frac{\partial J}{\partial x} f\right) = \min_{u\in u}\left(2x_1^2 + \frac{1}{2}u^2 + \frac{\partial J}{\partial x_1}x_2 + \frac{\partial J}{\partial x_2}u\right)$$

Since control law u is not constrained, then we can achieve

$$\frac{\partial H}{\partial u} = u + \frac{\partial J^0}{\partial x_2} = 0\text{, namely, } u = -\frac{\partial J^0}{\partial x_2}$$

We note that functional J is not related to time variable t, hence, $\frac{\partial J^0}{\partial t} = 0$. Then the following equation is always found:

$$2x_1^2 + \frac{1}{2}\left(\frac{\partial J}{\partial x_2}\right)^2 + \frac{\partial J}{\partial x_1}x_2 - \left(\frac{\partial J}{\partial x_2}\right)^2 = 0$$

In order to solve this partial differential equation, we may assume that the following optimal functional solution could satisfy above differential equation.

$$J^0 = a_1x_1^2 + 2a_2x_1x_2 + a_3x_2^2$$

The functional solution is substituted into above differential equation, and then we can know:

$$(1 - a_2^2)x_1^2 + (a_1 - 2a_2a_3)x_1x_2 + (a_2 - a_3^2)x_2^2 = 0$$

Then we can get the following equation group:

$$1 - a_2^2 = 0, \qquad a_1 - 2a_2a_3 = 0, \qquad a_2 - a_3^2 = 0$$

The solution of above algebraic equation group is the following:

$$a_2 = 1, \qquad a_3 = 1, \qquad a_2 = 2$$

The optimal solution of performance index functional J is the following:

$$J^0 = 2x_1^2 + 2x_1x_2 + x_2^2$$

Hence, optimal control law can achieve

$$u = -\frac{\partial J^0}{\partial x_2} = -2(x_1 + x_2)$$

This optimal control could be realized by state feedback component shown in figure7.5.

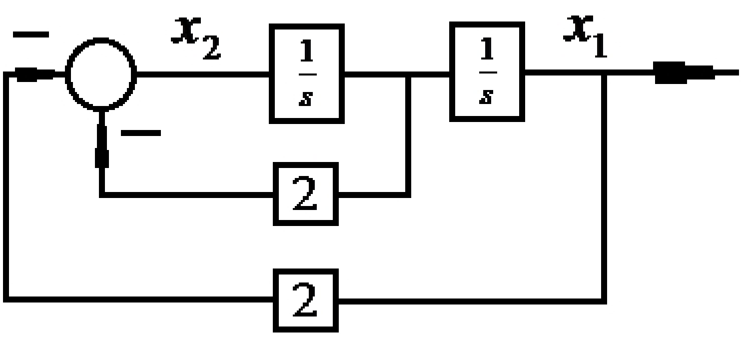

Figure7.5 State feedback block diagram of example7.25

In the following, we will explore the optimal state trajectory of system. The homogeneous state equation of system is the following

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x$$

Its solution can be gained as follows:

$$x^0(t) = e^{At}x(0) = L^{-1}\left[(sE-A)^{-1}\right]x(0)$$

$$= L^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1}\right\}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos t + \sin t) \\ -2e^{-t}\sin t \end{bmatrix}$$

Therefore, optimal control law $u^0(t)$ can be acquired:

$$u = -\frac{\partial J^0}{\partial x_2} = -2(x_1 + x_2) = 2e^{-t}(\sin t - \cos t)$$

Optimal performance index functional can be known as

$$J^0 = \int_0^\infty \left[2e^{-2t}(\sin t + \cos t)^2 + \frac{1}{2}\times 4e^{-2t}(\sin t - \cos t)^2\right]dt = \int_0^\infty 4e^{-2t}dt = 2$$

Example7.26 Assume the differential equation of system is the following:

$$\ddot{y} + 2\dot{y} + y = u$$

Please find optimal control u to make the following performance index functional J minimum; and the absolute value of control control u is always less than 1.

$$\min_u J = \int_0^\infty 1dt$$

Solution:

If we select $x_1 = y$ and $\dot{x}_1 = x_2 = \dot{y}$, then we can get the corresponding matrix equation of example7.26.

State equation: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Out equation: $$y = \begin{bmatrix} 1 & 0 \end{bmatrix}x$$

From Hamilton-Jacobi equation in equation(7.329), the following equation may be gained:

$$\frac{\partial J}{\partial t} = 0, \qquad L(x, \quad u, \quad t) = 1,$$

$$\left[\frac{\partial J}{\partial x}\right]^T f(x, \quad u, \quad t) = \begin{bmatrix} \frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2} \end{bmatrix}\begin{bmatrix} x_2 \\ -x_1 - 2x_2 + u \end{bmatrix} = \frac{\partial J}{\partial x_1}x_2 + \frac{\partial J}{\partial x_2}(-x_1 - 2x_2 + u)$$

$$-\frac{\partial J}{\partial t} = \min_{u\in U}\left\{1 + \begin{bmatrix} \frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2} \end{bmatrix}\begin{bmatrix} x_2 \\ -x_1 - 2x_2 + u \end{bmatrix}\right\} = 0$$

Then we consider the constraint condition: $-1 \le u \le 1$.

Calculate

$$\min_{u\in U}\left\{1+\begin{bmatrix}\dfrac{\partial J}{\partial x_1} & \dfrac{\partial J}{\partial x_2}\end{bmatrix}\begin{bmatrix}x_2 \\ -x_1-2x_2+u\end{bmatrix}\right\}=\min_{u\in U}\left\{\frac{\partial J}{\partial x_2}u\right\}$$

Obviously, optimal control $u^0(t)$ has to satisfy the following expression:

$$u^0=-\operatorname{sgn}\left(\frac{\partial J}{\partial x_2}\right)$$

Since functional J is not related to time t, Hamilton-Bellman equation may become

$$x_2\left[\frac{\partial J}{\partial x_1}-2\frac{\partial J}{\partial x_2}\right]-x_1\left[\frac{\partial J}{\partial x_2}\right]+\left[1-\left|\frac{\partial J}{\partial x_2}\right|\right]=0$$

This is a partial differential equation. We need to utilize computer software to acquire optimal functional J^0. And then employing derivative $\partial J/\partial x_2$ is to gain the optimal control u.

Summarizing the content of the section7.10.3, we can get the following steps of solving the optimal control problem on continuous dynamic programming.

1) Construct Hamilton function:

$$H(x,\ u,\ \lambda,\ t)=L(x,\ u,\ t)+\left[\frac{\partial J^0}{\partial x}\right]^T f(x,\ u,\ t)$$

2) Extreme of Hamilton is utilized to solve the optimal control u, namely

$$\frac{\partial H(x,\ u,\ t)}{\partial u}=0 \text{ (When control law } u \text{ is not constrained)}$$

Or,

$$\min_{u\in U} H(x,\ u,\ \lambda,\ t) \text{(When control law } u \text{ is permissible control)}$$

From these expression or equation, optimal control law u^0 can be solved, which is the function on variables x, t and $\partial J^0/\partial x$.

3) Optimal control u^0 is replaced into Hamilton-Bellman in equation(7.329). And then according to the boundary condition, optimal functional $J^0[x(t),\ t]$.

4) Optimal functional J^0 is inversely replaced into optimal control u^0 again. And then we can get optimal control function $u^0[x(t),\ t]$ which is the function of state variable. Hence, we can realize the closed-loop control in terms of optimal control and optimal performance index functional.

5) Optimal control $u^0[x(t),\ t]$ is substituted into system state equation, and then optimal system state trajectory $x^0(t)$ can be solved.

6) Optimal system state trajectory $x^0(t)$ can be substituted into performance index functional to gain optimal expression $J^0[x(t)]$.

However, what we need to point out is that optimal control strategy from Hamilton-Jacobi equation only represents necessary condition of optimization. In order to guarantee that performance index functional J has minimum, the matrix formed by second-order partial derivative of equation(7.331) to control law u has to be positive definite, namely,

$$\frac{\partial^2 L}{\partial u^2}+\frac{\partial}{\partial u}\left[\frac{\partial f^T}{\partial u}\bullet\frac{\partial J^0}{\partial x}\right]>0 \tag{7.344}$$

Inequality in equation(7.344) represents the sufficient condition that performance index functional J has minimum. This condition is called to Lerande(勒让得) condition.

In the general case, solving Hamilton-Bellman equation and Hamilton-Jacobi equation is very difficult, especially difficult for solving analytic solution. Commonly, computer is always employed to gain its numerical solution.

7.10.4 Relationship between minimum principle and dynamic programming

We postulate that system state equation is the following:

$$\dot{x}=f(x,\quad u,\quad t) \tag{7.345}$$

And performance index functional J

$$J=\int_{t_0}^{t_f} L[x(t)\quad u(t)\quad t]dt \tag{7.346}$$

In extreme principle, performance index functional is defined as the following Hamilton function in equation(7.347). Here, $\lambda(t)$ is adjoining vector or co-state vector.

$$H=L[x(t),\quad u(t),\quad t]+\lambda^T(t)f[x(t),\quad u(t),\quad t] \tag{7.347}$$

In order to make performance index function J get minimum, the Hamilton function should be absolutely minimum. However, for dynamic programming, Hamilton-Bellman equation is represented as follows:

$$-\frac{\partial J^0[x,\quad t]}{\partial t}=\min_{u\in U}\left\{L[x,\quad u,\quad t]+\left[\frac{\partial J^0[x,\quad t]}{\partial x}\right]^T f[x,\quad u,\quad t]\right\} \tag{7.348}$$

If functional expression $\left\{L+\frac{\partial J^0}{\partial x}f\right\}$ has minimum, relatively optimal expression will be acquired.

If order $\lambda(t)=\frac{\partial J^0[x(t),\quad t]}{\partial x}$, we easily know that the construction of dynamic programming and extreme principle is completely same. They can both calculate the same results on optimal control problem.

Chapter summary

This chapter focuses on the theme of optimal control which includes basic concept of optimal control, functional and its extreme condition, optimal control solution, Pontryagin's minimum principle, Bang-bang control, linear quadratic optimal control and its sub-optimal control, optimal control of discrete control system and dynamic programming. Firstly, some basic concepts such as function and functional are gradually provided and explained. Then, the solution of optimal control with unconstrained system is deduced and given out by variation method. Then, Pontryagin's minimum principle is deduced and explained to provide some examples for illustrating relative theories. On the basis of minimum principle, this chapter also illustrates Bang-bang control(switch control) and linear quadratic optimal control. Finally, optimal control of discrete control system is also deduced and explained to make readers easily understand given ideas. Besides these content, dynamic programming is also given out, which can deduce the same result as minimum principle. In this chapter, readers need to mainly master minimum principle, bang-bang control, linear quadratic optimal control and its sub-optimal control, optimal control of discrete system and dynamic programming.

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Review Questions

7.1. Please narrate the main thoughts of state feedback control system.

7.2. Please tell the specific application fields of output feedback control system.

7.3. What's pole assignment? How do you think the pole assignment of control is realized?

7.4. Please tell us the difference between pole assignment and partial pole assignment.

7.5 What is the Diophantine equation? What case is it fit for?

7.6 Please say out your comprehension on Pontryagin's minimum principle.

7.7 Please give out the performance functional of linear quadratic optimal control and relative Hamilton function.

7.8 Please list out and explain the necessary condition that makes performance functional get extreme value.

7.9 Please say the core principle of dynamic programming, and write out Bellman recursive equation of dynamic programming.

Problems

Problem7.1.Please find the minimum of the following function which range of variable is $[-2 \quad 6]$.

$$f(x)=x^3+4x^2+3x-5$$

Problem7.2.Please find the minimum of the following function.

$$f(x)=x^3+4x^2+3x-5$$

which constraint condition is $x^2+2x+3=0$.

Problem7.3.Function $f(x)=2x_1^2+3x_2^2+4x_3^2+2x_1x_2+3x_2x_3-5x_1x_3+4$; please find the extreme value points of given function and minimum.

Problem7.4.Please find the optimal curve of functional $J=J[x(t)]=\int_0^1 (x'^2+2tx+x)dt$. Its constraint conditions are $x(0)=0$ and $x(1)=1$.

Problem7.5.The state equation of a given system is