PID Control and Robust Control

This chapter will focus on two themes:PID control and Robust control. Firstly, the basic adjustment laws of PID (Proportional-Integral-Differential(Controller), PID) control are narrated and explored. Then relatively specific control laws deduced from PID controller will be provided for readers. On the basis of PID control, robust control will be introduced to realize the application of robust control in application.

Objectives

When you have finished this chapter, you should be able to:

- Understand the basic concepts on PID control and robust control
- Know the working principle of PID control and basic thoughts of robust control
- Describe robust control system correctly
- Recognize robust model and stability and performance.
- Utilize the sixteen-plant theorem to simply analyze a robust control system.

8.1 Introduction

In the control of dynamic systems, no controller has enjoyed the success and failure of the PID control.In all control design techniques, PID controller is the most widely used. Over 85% of all dynamic controllers are of the PID variety. There is actually a great variety of types and design methods for the PID controller.

What's a PID controller? The acronym PID stands for proportional-integral-differential control. Each of these, the P, the I and the D are terms in a control algorithm, and each has a special purpose. Sometimes certain of the terms are left out because they are not needed in the control design. This is possible to have a PI, PD or just a P control. It's very rare to have a ID control.

PID functions in control system are listed as follows:

1) Provide feedback for control system;
2) Eliminate steady state offsets through integral action;
3) Anticipate the future through derivative action;

PID control is an important ingredient of a distributed control system. The PID controllers are also embedded in many special-purpose control systems. In process control, more than 95% of the control loops are of PID type, most loops are actually PI control. Many useful features of PID control have not widely disseminated because they have been considered trade secrets.Typical examples are techniques for mode switches and anti-windup.

PID control is often combined with logic, sequential machines, selectors, and simple function blocks to build the complicated automation systems used for energy production, transportation, and manufacturing. Many sophisticated control strategies, such as mode predictive control, are also organized hierarchically. PID control is employed at the lowest control; the multi-variable controller gives the setpoints to the controllers at the lower control. The PID controller is an important component in almost every control system.

This chapter gives an introduction to PID control and robust control. The basic algorithm and various representations are presented in detail. A description of the properties of the controller in a closed loop based on intuitive arguments is given. The phenomenon of reset windup, which occurs when a controller with integral action is connected to a process with a saturating actuator, is discussed, including several methods to avoid it.

8.2 PID Structure

8.2.1 PID problem

The standard PID control configuration is shown in figure8.1.It is sometimes called to the “PID” parameter form.

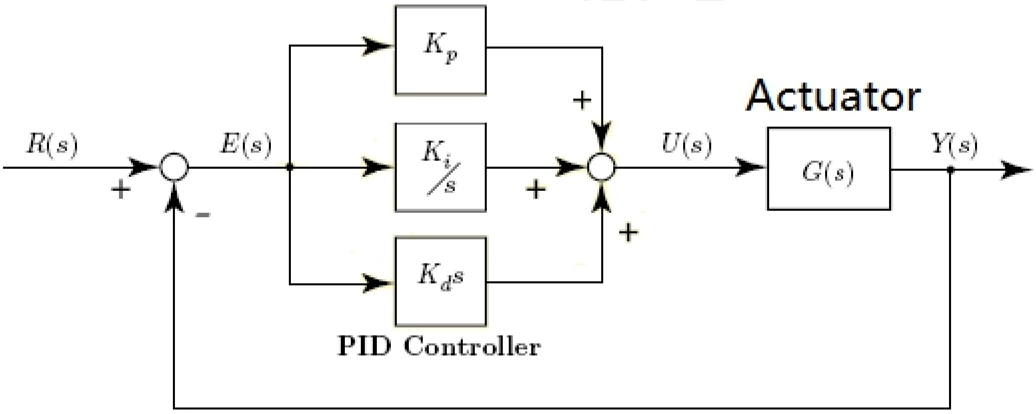


Figure8.1 PID control systems

In such configuration, control signal $u(t)$ is the sum of three terms. Each of these terms is a function of the tracking error $e(t)$. The term K_p indicates that this term is proportional to the error. The term K_i/s is an integral term, and the term $K_d s$ is a derivative term. Each of these terms can work independently to the other.

8.2.2 PID control structure

The combination of the proportional, integral,derivative actions can be done in different ways. In the so-called ideal or non-interaction form, the PID controller is described by the following function:

Proportional-integral-derivative function:

$$u(t)=K_P\left(e+\frac{1}{T_I}\int_0^t edt+T_D\frac{de}{dt}\right)=K_Pe+K_I\int_0^t edt+K_D\frac{de}{dt} \tag{8.1}$$

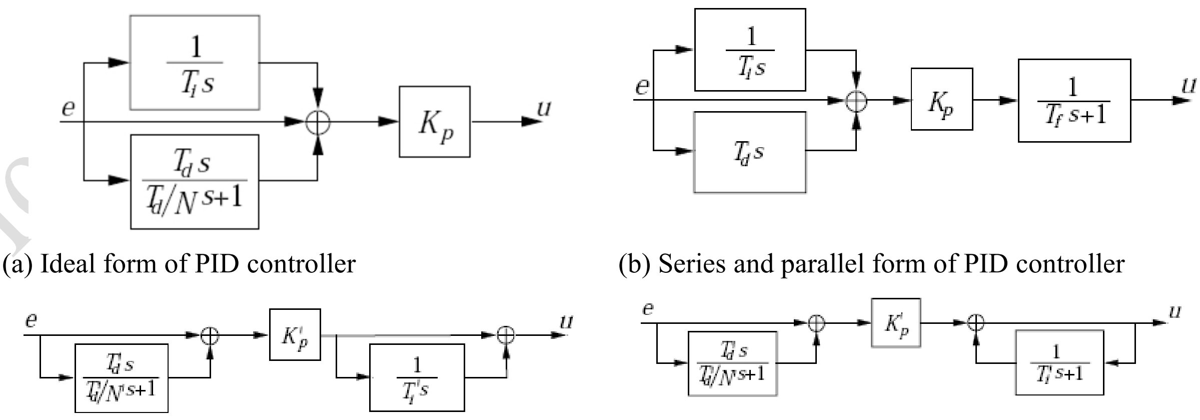
Transfer function when all initial conditions are zero:

$$U(s)=\left(K_P+\frac{K_I}{s}+K_Ds\right)E(s) \tag{8.2}$$

Here, K_P is the proportional gain, T_I is the integral time constant, and T_D is derivative time constant. Integral constant K_I can be calculated by expression $K_I=\frac{K_P}{T_I}$; derivative constant K_D may be done by expression $K_D=K_PT_D$.

From equation(8.2), it is known that the changing relation between input variable $E(s)$ and output variable $U(s)$ is up to the proportional amplification coefficient, integral time constant and derivative time constant.

When there is no integral component in PID controller, PID control is converted into PD control; when derivative part is zero, PID control is converted into PI control; and when integral component and derivative component are both zero, PID control is completely converted into P control. If amplification coefficient K_P become larger, the role of proportional part will become stronger. However, when amplification coefficient K_P exceeds the permissible range of PID control, control system will appear oscillating phenomenon, even divergent oscillation. If system is excessively stable, redundant error is easily increased in control system. Integral part is used to eliminate static error of system, which is decided by integral time constant T_I .If it become bigger and bigger, the integral role will become weaker and weaker. Derivative part is used to suitably adjust the changing velocity of error, overcome external disturbance, and prevent the increment of deviation. If derivative time constant T_D is very large, there are easily a larger oscillation to controlled variable in control system. That's to say, PID control is in nature to confirm three parameters:the proportional gain K_P, integral constant K_I and derivative constant K_D.



(a) Ideal form of PID controller

(b) Series and parallel form of PID controller

(c)Feedforward form of PID controller

(c) Feedback and feedforward form

Figure8.2 Relative forms of PID controller

Besides the block diagram in figure8.1, the following 4 forms are used in practice, shown in figure8.2.

1.P(proportional,P) control

The proportional control action is proportional to the current control error according to the following expression:

$$u(t)=K_P e(t)=K_P(r(t)-y(t)) \tag{8.3}$$

Here, sign K_P is the proportional gain. Its meaning is straightforward, since it implements the typical operation of increasing the control variable when the control error is large. The transfer function of a proportional controller can be derived trivially as

$$G(s)=K_P \tag{8.4}$$

With regard to the on-off controller, a proportional controller has the advantage of providing a small control variable when the control error is small and therefore to avoid excessive control efforts. The main drawback of using a pure proportional controller is that it produces a steady-state error. It's worth noting that this occurs even if the process presents an integrating dynamics, in case a constant load disturbance occurs. This motivates the addition of a bias or reset term u_b, namely,

$$u(t)=K_P e(t)+u_b \tag{8.5}$$

The value of u_b can be fixed at a constant level(usually at $(u_{\max}+u_{\min})/2$) or can be adjusted manually until the steady-state error is reduced to zero. It is worth noting that in commercial products the proportional gain is often replaced by the proportional band PB, that's the range of error that causes a full range change of the control variable.

$$PB=\frac{100}{K_P} \tag{8.6}$$

2.I(integral,I) control

The integral action is proportional to the integral of control error, which can be expressed as

$$u(t)=K_I\int_0^t e(\tau)d\tau \tag{8.7}$$

Here, sign K_I is integral control gain. It appears that the integral action is related to the past value of the control error. The corresponding transfer function is

$$G(s)=\frac{K_I}{s} \tag{8.8}$$

The presence of a pole at the origin of the complex plane allows the reduction to zero of the steady-state error when a step reference signal is applied or a step load disturbance occurs. In other words, the integral action is able to set automatically the correct value of reset term u_b so that the steady-state error is zero. That's to say, the main function of integral action is to make sure that the process output agrees with the set-point in steady state. With proportional control, there is normally a control error in steady state. With integral action, a small positive error will always lead to an increasing control signal, and a negative error will give a decreasing control signal no matter how small the error is.

The following simple example shows that the steady-state error will always be zero with integral action. We postulate that the system is in steady state with a constant control signal(u_0) and a constant error$\left(e_0\right)$.From equation(8.1), the control signal can be given by

$$u(t)=K_P\left(e+\frac{1}{T_I}\int_0^t edt\right) \tag{8.9}$$

The form of transfer function is gained by

$$U(s)=K_P\left(1+\frac{1}{T_I s}\right) \tag{8.9}$$

As long as $e_0 \neq 0$, this clearly contradicts the assumption that the control signal u_0 is constant. A controller with integral action will always give zero steady-state error.

Integral action can also be visualized as a device that automatically resets the bias term u_b of a proportional controller. This is illustrated in the block diagram in figure8.3, which shows a proportional controller with a reset that is adjusted automatically. The adjustment is made by feeding back a signal, which is a filtered value of the output, to the summing point of the controller. This was actually one of the early inventions of integral action, or "automatic reset", as it was so called.

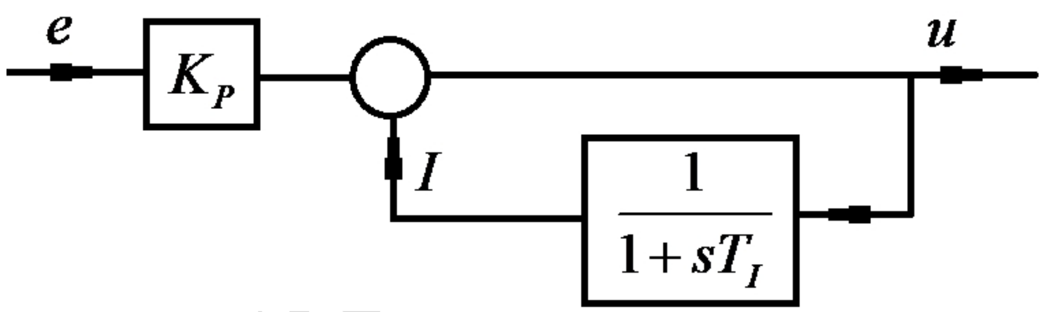


Figure8.3 Implementation of integral action as positive feedback around a lag

The following equations follow from the block diagram in figure8.3:

$$u=K_P e+I \tag{8.10a}$$

$$T_I\frac{dI}{dt}+I=u \tag{8.10b}$$

After eliminating output variable u, there is the following equation:

$$T_I\frac{dI}{dt}+I=K_P e+I$$

Hence,

$$T_I\frac{dI}{dt}=K_P e \tag{8.11}$$

Equation(8.11) shows that controller in figure8.3 is PI controller in fact.

3.D(Derivative, D) control

While the proportional action is based on the current value of the control error and the integral action is based on the past values of the control error, the derivative action is based on the predicted future values of the control error. An ideal derivative control law can be expressed as

$$u(t) = K_D \frac{de(t)}{dt} \tag{8.12}$$

Here, sign K_D is derivative gain. The corresponding controller transfer function is:

$$G(s) = K_D s \tag{8.13}$$

In order to understand better the meaning of the derivative action, it is worth considering the first two terms of the Taylor series expansion of the control error at time T_D ahead:

$$e(t+T_D) \approx e(t) + T_D \frac{de(t)}{dt} \tag{8.14}$$

The control signal is thus proportional to an estimation of the control error at time T_D ahead, where the estimation is obtained by linear extrapolation.

If a control law proportional to expression in equation(8.14) is considered,

$$u(t) = K_P \left(e + T_D \frac{de(t)}{dt} \right) \tag{8.15}$$

thus this naturally results in a PD controller.The control variable at time t is therefore based on the predicted value of the control error at time $(t+T_d)$. For this reason the derivative action is also called anticipatory control, or rate action.

Obviously, the derivative action has a great potentiality in improving the control performance as it can anticipate an incorrect trend of the control error and counteract for it.Of course, there are all types of PID controllers in industries.

8.3 Control System With PID controller

8.3.1 System characteristic equation with PID controller

The general transfer function of PID controller is as follows:

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) = K_p + \frac{K_i}{s} + K_d s = K_p \frac{(as+1)(bs+1)}{s} \tag{8.16}$$

Here,

$$K_i = \frac{K_p}{T_i}, \qquad K_d = K_p T_d, \qquad ab = T_i T_d, \qquad a+b = T_i$$

Hence,PID control component increases the number of zeros by two and the number of pole by ones, where the two zeros are respectively located at $s=-1/a$ and $s=-1/b$ and the pole is located at $s=0$. PID controller is designed by properly choosing the parameters such as K_p, K_i or K_d to control given system with all advantages of the PD and PI controllers combined.

A typical closed-loop control system is shown with PID controller in figure8.4.

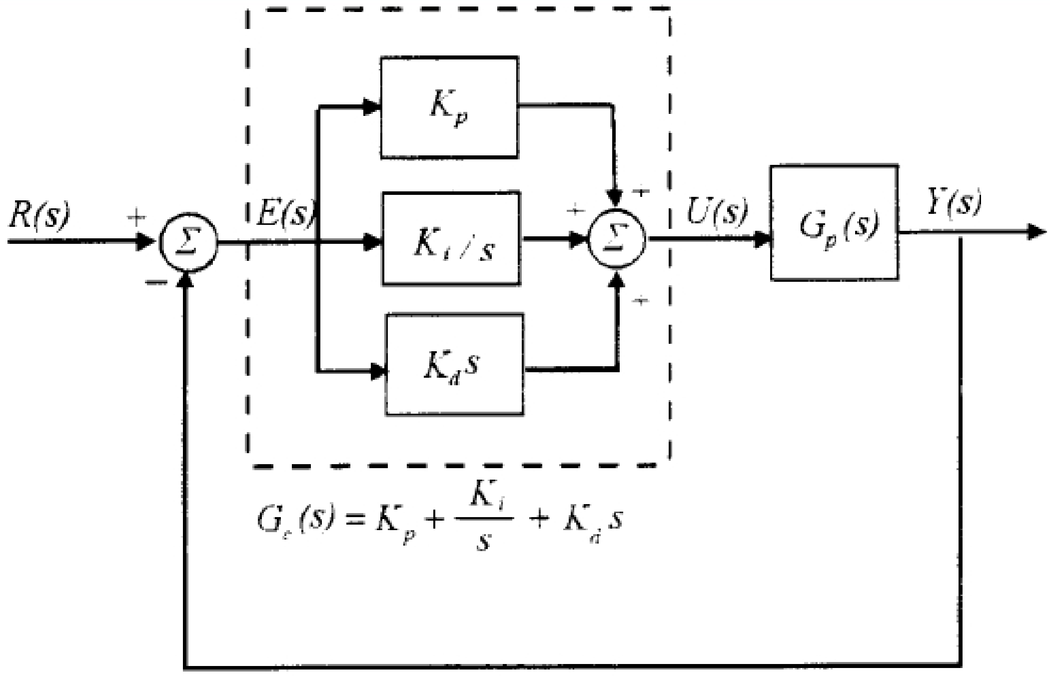


Figure8.4 A typical control system with PID controller

From figure8.4, the closed-loop control system transfer function can be gained as

$$\phi(s)=\frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \tag{8.16}$$

Hence, for unit negative feedback system, its characteristic equation yields to

$$1+G_c(s)G_p(s)=0 \tag{8.17}$$

If feedback component is $H(s)$, not a unit feedback, corresponding characteristic equation can be written out similarly as

$$1+G_c(s)G_p(s)H(s)=0 \tag{8.18}$$

In terms of Routh's stability theorem, so long as the roots of equation(8.17) and equation(8.18) are located in left plane or unit circle of complex plane, the control system with PID controller will be stable.

Example8.1 Assume a control system with PID controller is shown in figure8.4. The transfer function of plant component is the following:

$$G(s)=\frac{K}{(1+sT_1)(1+sT_2)}$$

Using PID controller is to achieve arbitrary pole placement and to simultaneously drive the position steady-error to zero.

Solution

There includes PID controller in figure8.4; hence in terms of equation(8.17), we can easily write out the following characteristic equation of closed-loop control system:

$$1+K_p\left(1+\frac{1}{T_i s}+T_d s\right)\frac{K}{(1+sT_1)(1+sT_2)}=0$$

Namely,

$$s^3+\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{KK_pT_d}{T_1T_2}\right)s^2+\left(\frac{1}{T_1T_2}+\frac{KK_p}{T_1T_2}\right)s+\frac{KK_p}{T_1T_2T_i}=0$$

Hence, this system is a third-order control system. The general form of the characteristic equation of third-order system is as follows:

$$(s+\tau\omega)\left(s^2+2\varsigma\omega_n s+\omega_n^2\right)=0$$

Equating coefficient of like powers of complex variable s in the last two characteristic equations, and after some algebraic manipulations, we arrive at the following values for the PID controller parameters K_p, T_i and T_d:

$$K_p=\frac{T_1T_2\omega_n^2(1+2\zeta\tau)-1}{K},\quad T_i=\frac{T_1T_2\omega_n^2(1+2\zeta\tau)-1}{T_1T_2\tau\omega_n^3},\quad T_d=\frac{T_1T_2\omega_n(\tau+2\zeta)-T_1-T_2}{T_1T_2\omega(1+2\tau)-1}$$

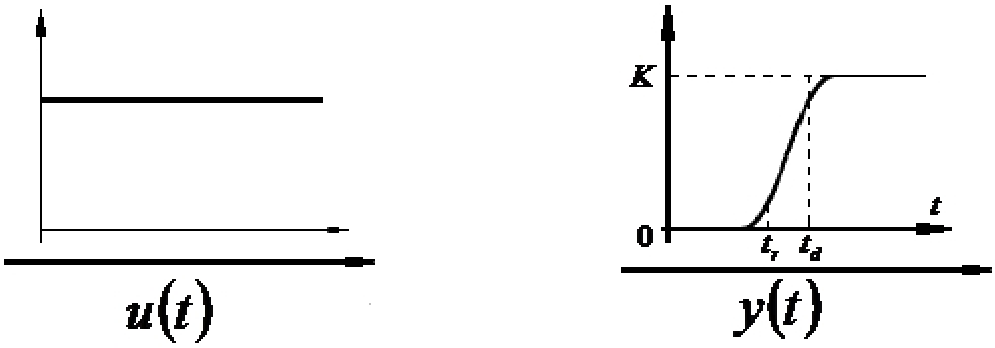
Hence, for any desired values of coefficients ζ, ω_n and τ, we can always find adjustment parameters K_p, T_i and T_d such that the closed-loop characteristic polynomial has the desired form $(s+\tau\omega)\left(s^2+2\varsigma\omega_n s+\omega_n^2\right)$. Since the roots of this polynomial are the poles of the closed-loop system, it follows that we can achieve arbitrary pole placement of the given second-order system with a PID controller. This result is of paramount importance, since controlling the poles one controls completely the stability and may greatly influence the time response of the closed-loop system.

8.3.2 Design of PID controller using the Ziegler-Nichols method

The PID controller has the flexibility of simultaneously tuning three parameters: K_p, T_i and T_d. This allows a PID controller to satisfy the design requirements in many practical cases, a fact which makes the PID controller the most frequently met controller in practice. Then in classical control theory, it is known that the appropriate values of the parameters K_p, T_i and T_d of the PID controller may be chosen by trial and error.To facilitate the determination of the appropriate values of the parameter K_p, T_i and T_d, even for cases where a mathematical model for the system under control is not available, Ziegler and Nichols have suggested the following two rather simple and practically useful methods.

1. Transient response method

In this case, the system under control is excited by the unit step function(Figure8.5a). The shape of the transient response of the open-loop system may have the general form shown in figure8.5b.



(a) Input signal (b) Output signal

Figure8.5 Transient response method

In this case, we introduce two parameters: delay time t_d and rising time t_r. Aiming in achieving a damping ratio ζ of about 0.2(which correspond to the overshoot of about 25%), the value of parameters K_p, T_i and T_d of the PID controller are chosen according to table8.1.

Table8.1 The Value of the Parameters K_p, T_i and T_d Using the Ziegler-Nichols Transient Response Method

Controller		K_p	T_i	T_d
Proportional	P	t_r/t_d	∞	0
Proportional-Integral	PI	$0.9t_r/t_d$	$t_d/0.3$	0
Proportional-Integral-Derivative	PID	$1.2t_r/t_d$	$2t_d$	$0.5t_d$

It is useful that to mention that here the transfer function of the control system may be appropriated as follows:

$$G(s)=K\left(\frac{e^{-t_d s}}{1+t_r s}\right) \tag{8.19}$$

Furthermore, upon using table8.1, the PID controller transfer function $G_c(s)$ becomes

$$G_c(s)=K_p\left(1+\frac{1}{T_i s}+T_d s\right)=1.2\frac{t_r}{t_d}\left(1+\frac{1}{2t_d s}+0.5t_d s\right)=0.6t_r\left[\frac{(s+1/t_d)^2}{s}\right] \tag{8.20}$$

Namely, the PID controller has a pole at original point and a double zeros at $s=-1/t_d$.

Example8.2 In figure8.5, the unit step response of an executor is given. Transient response parameters are $t_d=150\sec$ and $t_r=75\sec$. Please use table8.1 to determine the parameters K_p, T_i and T_d of the PID controller.

Solution

According to equation(8.20) and table8.1, we readily have

$$K_p = 1.2\frac{t_r}{t_d} = 1.2\times\frac{75}{150} = 0.6$$

$$T_i = 2t_d = 2\times150 = 300\,\text{sec}$$

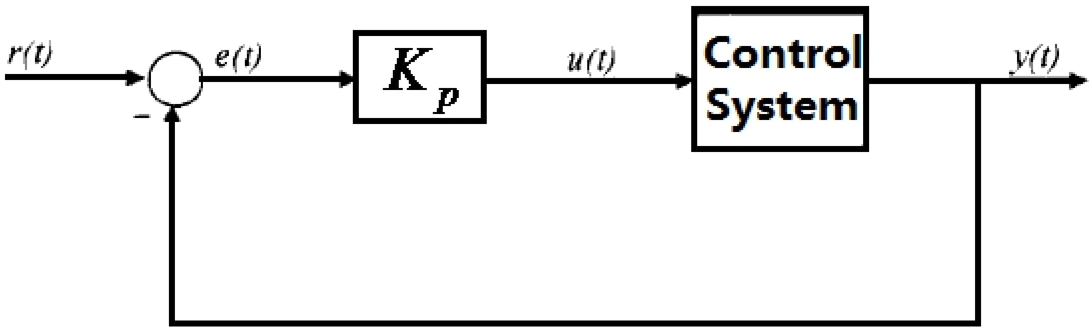
$$T_d = 0.5t_d = 0.5\times150 = 75\,\text{sec}$$

Hence, the transfer function of PID controller is as follows:

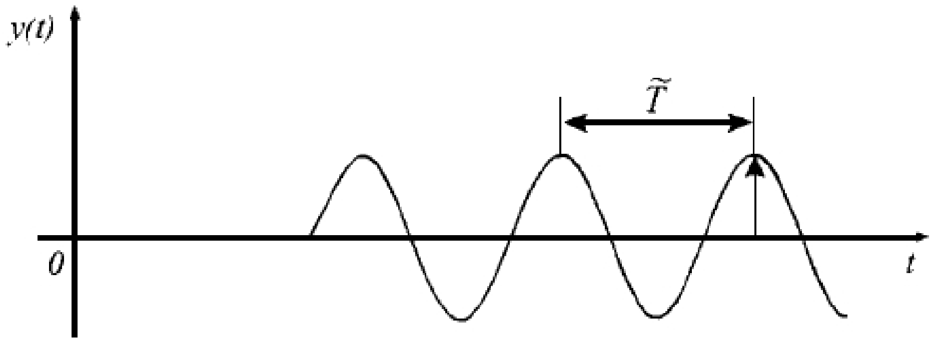
$$G_c(s) = 0.6\left(1+\frac{1}{300s}+75s\right) = 45\left[\frac{(s+1/150)^2}{s}\right]$$

2. The stability limit method

Here, we start by controlling the system only with the proportional controller shown in figure8.6a. The gain K_p is slowly increased until a persistent oscillation is reached in figure8.6b. At this point, we mark down the value of parameter K_p, denoted as $\tilde{K}_p$ as well as the value of respective oscillation period, denoted as $\tilde{T}$. Then the parameters of K_p, T_i and T_d of the PID controller are chosen in table8.2.



(a) Closed-loop control system block diagram with proportional controller



(b) Sustained oscillations with period $\tilde{T}$

Figure8.6 The stability method

For such case, and upon using table8.2, the transfer function of PID controller becomes

$$G_c(s)=K_p\left(1+\frac{1}{T_i s}+T_d s\right)=0.6\tilde{K}_p\left(1+\frac{1}{0.5\tilde{T}s}+\frac{\tilde{T}}{8}s\right)=\left(0.075\tilde{K}_p\tilde{T}\right)\left[\frac{\left(s+4/\tilde{T}\right)^2}{s}\right] \quad (8.21)$$

Table8.2 The Value of the Parameters K_p, T_i and T_d Using the Ziegler-Nichols Stability Limit Method

Controller		K_p	T_i	T_d
Proportional	P	$0.5\tilde{K}_p$	∞	0
Proportional-Integral	PI	$0.45\tilde{K}_p$	$\tilde{T}/1.2$	0
Proportional-Integral-Derivative	PID	$0.6\tilde{K}_p$	$\tilde{T}/2$	$\tilde{T}/8$

That's to say, the PID controller has a pole at original point and a double zeros at $s=-4/\tilde{T}$.

Example8.3 In figure8.6, the block diagram of control system is given. Please use Ziegler-Nichols Stability Limit method to determine the parameters of PID controller. And used Routh stability criterion to calculate the values of $\tilde{K}_p$ and $\tilde{T}$. The transfer function of control system is the following:

$$G(s)=\frac{1}{s(s+1)(s+4)}$$

Solution

For there is only one proportional component in figure8.6, the transfer function of closed-loop transfer function can be given as follows:

$$\frac{Y(s)}{R(s)}=\frac{K_p}{s(s+1)(s+4)+K_p}$$

The proportional value K_p of PID controller that make control system marginally stable, in which case sustained oscillations occur, can be obtained using Routh's stability criterion. The characteristic equation of control system is given as follows:

$$s^3+5s^2+4s+K_p=0$$

The Routh array is as follows:

$$\begin{array}{lll} s^3 & 1 & 4 \\ s^2 & 5 & K_p \\ s^1 & \dfrac{20-K_p}{5} & \\ s^0 & K_p & \end{array}$$

After examining the coefficient of the first column of above Routh table, we readily find that sustained oscillation can occur when $K_p = 20$. Thus, critical gain $\tilde{K}_p$ is 20.

With the gain K_p set equal to $\tilde{K}_p$, the characteristic equation becomes

$$s^3 + 5s^2 + 4s + 20 = 0$$

Substituting $s = j\omega$ into the characteristic equation, we can get the frequency of sustained oscillation. We have

$$(j\omega)^3 + 5(j\omega)^2 + 4(j\omega) + 20 = 0$$

Hence, in terms of complex rules, we can order that real and imagine are zero, and then we can gain the solution: $\omega = 2$.

The period $\tilde{T}$ of sustained oscillation can be calculated out as follows:

$$\tilde{T} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi = 3.14$$

Referring table8.2, we may determine the parameters K_p, T_i and T_d of the PID controller:

$$K_p = 0.6\tilde{K}_p = 0.6 \times 20 = 12$$

$$T_i = 0.5\tilde{T} = 0.5 \times 3.14 = 1.57$$

$$T_d = 0.125\tilde{T} = 0.125 \times 3.14 = 0.3925$$

Hence, the transfer function of PID controller in figure8.6 can be achieved as follows:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = 12\left(1 + \frac{1}{1.57s} + 0.3925s\right) = 4.71\left[\frac{(s+1.27389)^2}{s}\right]$$

The PID controller has a pole at original point and a double zeros at $s = -1.27389$. The closed-loop transfer function in figure8.6 can be given as follows:

$$\phi(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{\left[\frac{4.71(s+1.27389)^2}{s}\right]\left[\frac{1}{s(s+1)(s+4)}\right]}{1+\left[\frac{4.71(s+1.27389)^2}{s}\right]\left[\frac{1}{s(s+1)(s+4)}\right]}$$

$$= \frac{4.71s^2 + 12s + 7.643}{s^4 + 5s^3 + 8.71s^2 + 12s + 7.643}$$

The unit step response $Y(s)$ of closed-loop control system can be acquired by

$$Y(s) = \phi(s)R(s) = \left[\frac{4.71s^2 + 12s + 7.643}{s^4 + 5s^3 + 8.71s^2 + 12s + 7.643}\right]\left[\frac{1}{s}\right]$$

In this control system in figure8.6, if the overshoot is excessive, it can be reduced by tuning the controller parameters.

The design of a PID controller in a control system consists of the following reference steps:

1.Evaluate the performance of the uncompensated system to determine how much improvement in transient response is required.

2.Design the PID controller to meet the transient response specifications. The design includes the zero location and the loop gain.

3.Simulate the system to be sure all requirements have been met.

4.Redesign if the simulation shows that requirements have not been met.

5.Design the PID controller to yield the required steady-state error.

6.Determine the parameters K_p, T_i and T_d of the PID controller in figure8.4.

7.Simulate the system to be sure all requirements have been met.

8.Redesign if simulation shows that requirements have not been met.

8.3.3 Digital implement of PID control

If a digital implement of PID controller is adopted, then previous control laws considered have to be discretized. This can be done with any of the available discretization method. Now let's consider the continuous time expression of a PID controller in ideal form:

$$u(t) = K_p\left[e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right] \tag{8.22}$$

And we define the sampling time Δt . The integral term in equation(8.22) can be approximately by using backward finite differences as

$$\int_0^{t_k} e(\tau)d\tau = \sum_{i=1}^{k} e(t_i)\Delta t \tag{8.23}$$

Here, sign $e(t_i)$ is the error of the continuous time system at the ith sampling instant. By applying the backward finite difference also into the derivative term, it results in

$$\frac{de(t_k)}{dt} = \frac{e(t_k) - e(t_{k-1})}{\Delta t} \tag{8.24}$$

Then, discrete control law becomes

$$u(t_k) = K_p\left[e(t_k) + \frac{\Delta t}{T_i}\sum_{i=1}^{k} e(t_i) + \frac{T_d}{\Delta t}(e(t_k) - e(t_{k-1}))\right] \tag{8.25}$$

In this way, the value of control variable can be determined directly. Alternatively, the control variable at the time instant t_k can be calculated on the basis of the value $u(t_{k-1})$ at the previous time instant t_{k-1}. By subtracting the expression of $u(t_{k-1})$ from that of $u(t_k)$, we can obtain the following expression:

$$u(t_{k-1}) = K_p\left[e(t_{k-1}) + \frac{\Delta t}{T_i}\sum_{i=1}^{k-1} e(t_i) + \frac{T_d}{\Delta t}\left(e(t_{k-1}) - e(t_{k-2})\right)\right] \tag{8.26}$$

$$\begin{aligned} u(t_k) - u(t_{k-1}) &= K_p\left[e(t_k) + \frac{\Delta t}{T_i}e(t_k) + \frac{T_d}{\Delta t}e(t_k) - 2\frac{T_d}{\Delta t}e(t_{k-1}) - e(t_{k-1}) + \frac{T_d}{\Delta t}e(t_{k-2})\right] \\ &= K_p\left[\left(1 + \frac{\Delta t}{T_i} + \frac{T_d}{\Delta t}\right)e(t_k) + \left(-1 - 2\frac{T_d}{\Delta t}\right)e(t_{k-1}) + \frac{T_d}{\Delta t}e(t_{k-2})\right] \end{aligned} \tag{8.27}$$

The algorithm in equation(8.27) is called incremental algorithm or velocity algorithm, while algorithm in equation(8.25) is called position algorithm. Expression(8.27) can be written more compactly as

$$u(t_k) - u(t_{k-1}) = K_1 e(t_k) + K_2 e(t_{k-1}) + K_3 e(t_{k-2}) \tag{8.28}$$

Here,

$$\begin{aligned} K_1 &= K_p\left(1 + \frac{\Delta t}{T_i} + \frac{T_d}{\Delta t}\right) \\ K_2 &= K_p\left(-1 - 2\frac{T_d}{\Delta t}\right) \\ K_3 &= K_p\frac{T_d}{\Delta t} \end{aligned} \tag{8.29}$$

Here, we define D^{-1} as the backward shift operator such as

$$D^{-1}u(t_k) = u(t_{k-1}) \tag{8.30}$$

Then the discrete PID controller in velocity form of equation(8.28) can be expressed as

$$u(t_k) = \frac{K_1 + K_2 D^{-1} + K_3 D^{-2}}{1 - D^{-1}} e(t_k) \tag{8.31}$$

In equation(8.31), coefficients K_1, K_2 and K_3 can be viewed as the tuning parameters.

On information on tuning parameters of PID controllers, readers can consult related PID control books published in market. Here, author will not narrate and explain this theme in detail.

8.4 Lead-lag Compensator

Closely related to PID control is the idea of lead-lag compensation. These ideas are frequently used in practice, specially when compensators are built with electronic hardware components. The transfer function of these compensators is the form:

$$C(s) = \frac{\tau_1 s + 1}{\tau_2 s + 1} \tag{8.32}$$

If $\tau_1 > \tau_2$, then this is a lead component and when $\tau_1 < \tau_2$, this is a lag component.

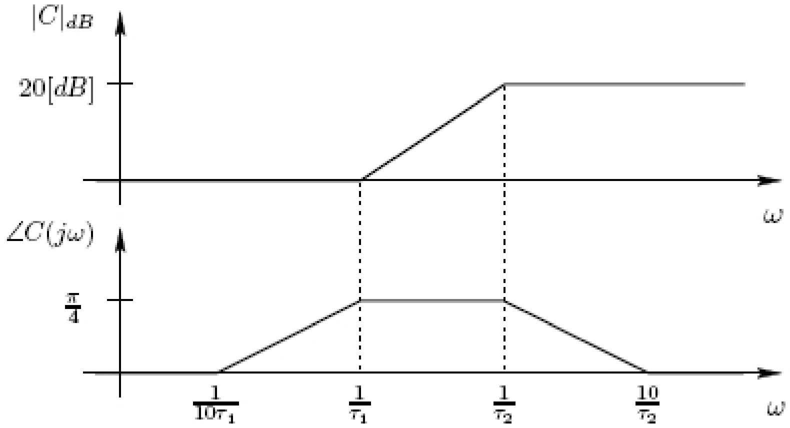


Figure8.7 Approximate Bode diagram of lead component($\tau_1 = 10\tau_2$)

The straight line approximation to the Bode diagram for Lead component is given in figure8.7. The lead compensator acts like an approximate derivation. From figure8.7, we see that this graph produces approximate 45° phase advance at $\omega = 1/\tau_1$ without a significant change in gain. Thus if we have a simple feedback loop which passes through the -1 point at frequency ω_1, then inserting a lead compensator such that $\omega_1 \tau_1 = 1$ will give a 45° phase margin. Of course, the disadvantage is an increase in the high frequency gain which can amplify high frequency noise.

An alternative interpretation of a lead component is obtained on considering its pole-zero structure. From equation(8.32) and the fact that $\tau_1 > \tau_2$, we see that it introduces a pole-zero pair, where the zero(at $s = -1/\tau_1$) is significantly closer to the imaginary axis than the pole that is located at $s = -1/\tau_2$.

On the other hand, the straight line approximation to the Bode diagram for Lag component is given in figure8.8.

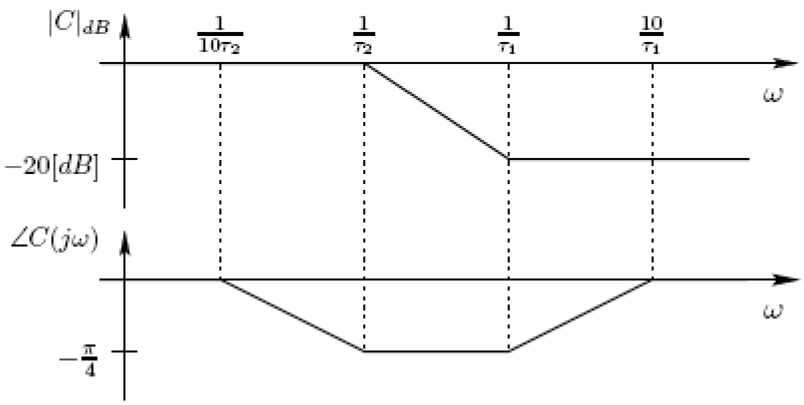


Figure8.8 Approximate Bode diagram of lag component($\tau_2 = 10\tau_1$)

From figure8.8, we see that the low frequency gain is 20[dB] higher than the gain at

$\omega = 1/\tau_1$. Thus this circuit, when used in a feedback loop, gives better low frequency tracking and disturbance rejection. A disadvantage of this circuit is the additional phase lag experienced between $1/10\tau_2$ and $10/\tau_1$.

Hence, $1/\tau_2$ is typically chosen to be smaller than other significant dynamics in the plant. From the point of view of its pole-zero configuration, the lag component introduces a pole located at $s = -1/\tau_2$, which is significantly closer to the imaginary axis than the zero located at $s = -1/\tau_1$.

In chief, the lead component feature used in design is its phase advance characteristics. The lag component instead is useful due to its gain characteristic at low frequency.

When both effects are required, a lead-lag effect can be obtained by cascading a lead and a lag compensator.

8.5 Basic Thoughts of Robust Control

We design some control system to reach predicted performance , but there are always some unsatisfied performance for some uncertainty factors exist in control system. In order to gain satisfying performance with uncertain factors, people put forward robust control. Designing highly accurate systems in the presence of significant uncertainty is a classical feedback design problem. The theoretical bases for the solution of this problem date back to the works of H.S.Black and H.W.Bode in the early 1930s, when this problem was referred to as the sensitivities design problem. A significant amount of literature has been published since then regarding the design of systems subject to large process uncertainty. The designer seeks to obtain a system that performs adequately over a large range of uncertain parameters. *A system is said to be robust when its is durable, hardy, and resilient*.

Robust control system has the following characteristics:

(1) Low sensitivity;

(2) Keep stable over the range of parameter variations;

(3) The performance continues to meet the specifications in the presence of a set of changes in the system parameters.

Robustness is the low sensitivity to effects that are not considered in the analysis and design phase such as disturbances, measurement noise, and unmodeled dynamics. The system should be able to withstand these neglected effects when performing the tasks for which is designed.

For any control system, all factors are divided into two types: certainty and uncertainty. An example is given to demonstrate basic thoughts of robust control.

Example8.4 The mass of car is marked as M , road friction is expressed as sign μ . Dynamic model of car is shown in figure8.9. Motion equation of car is as follows:

$$M\frac{d\upsilon}{dt} + \mu\upsilon = f$$

If the mass of car changes with change of car load and if friction coefficient μ changes with the change of road situation, coefficients in above equation have a certain uncertainty. The accurate value of mass M and friction coefficient μ can not be gained.

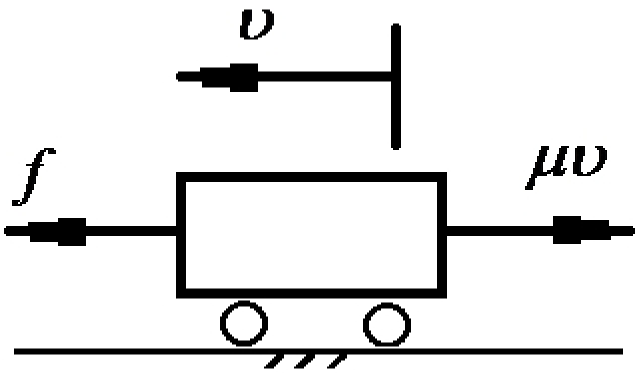


Figure8.9 Car speed control model

Assume that the change range of parameters is given as follows:

$$M_0 - \delta_1 \le M \le M_0 + \delta_1$$

$$\mu_0 - \delta_2 \le \mu \le \mu_0 + \delta_2$$

Here, δ_1 and δ_2 are given constant. Then actual system could be depicted as

$$(M_0 + \Delta M)\frac{d\upsilon}{dt} + (\mu_0 + \Delta\mu)\upsilon = f, \quad |\Delta M| \le \delta_1, \quad |\Delta\mu| \le \delta_2$$

Hence, certainty part in control system can be expressed as

$$M_0 \frac{d\upsilon}{dt} + \mu_0 \upsilon = f$$

And uncertainty part in control system can be depicted into parameter vector set.

$$\{(\Delta M, \ \Delta\mu) \,|\, |\Delta M| \le \delta_1, \ |\Delta\mu| \le \delta_2\}$$

If transfer function f to car speed υ is used to describe speed control model in figure8.9, we have

$$G(s) = \frac{1}{Ms + \mu} = G_0(s) + \Delta G(s)$$

Here,

$$\begin{cases} G_0(s) = \dfrac{1}{M_0 s + \mu_0} \\ \Delta G(s) = \dfrac{1}{Ms + \mu} - \dfrac{1}{M_0 s + \mu_0} = -\dfrac{\Delta M s + \Delta\mu}{(M_0 s + \mu_0)[(M_0 + \Delta M)s + (\mu_0 + \Delta\mu_0)]} \end{cases}$$

Obviously, there easily find a boundary function $r(j\omega)$ which meets uncertainty parameter vector set, we can gain

$$|\Delta G(j\omega)| \le |r(j\omega)|$$

Hence, in frequency domain, when control system is described, this system in figure8.9 can be expressed as

$$\sum\nolimits_0: \quad G_0(s)$$
$$\Delta\sum: \quad \{\Delta G(s) \| \Delta G(j\omega)| \le |r(j\omega)|, \quad \forall_\omega \in R\}$$

Above example demonstrates that given system structure is very definite, and that uncertainty is depicted by coefficient boundary perturbation of differential equation. Uncertainty set may be boundary set of parameter space, or rational function set. For this example, whatever way is used to depict given system is up to expression form of certainty system. In practice, a lot of systems don't perhaps gain accurate information on structure.

8.6 Uncertainty Representation of Robust Control System

In spite of robustness analysis or robust controller design, we should firstly build mathematical model of controlled object, namely nominal model $\sum_0$ and uncertainty set $\Delta\sum$. Nominal model $\sum_0$ could be acquired by logic deduction or system identification. Uncertainty representation will be explored in this section.

Uncertainty in control systems may stem from different sources. Model uncertainty is one main consideration. Other considerations include sensor and actuator failures, physical constraints, changes in control purposes, loop opening and loop closure, etc. Moreover, in control design problems based on on optimization, robustness issues due to mathematical objective functions not properly describing the real control problem may occur. On the other hand, numerical design algorithms may not be robust. However when we refer to robustness in this chapter, we mean robustness with respect to model uncertainty. We also assume that a fixed linear controller is used.

Model uncertainty may have several origins. In particular, it may be caused by the following points.

1. Parameters in a linear model, which are approximately known or are simply in error.
2. Parameters, which may vary due to nonlinearities or changes in operating conditions.
3. Neglected time delays and diffusion process
4. Imperfect measurement devices
5. Reduced(low-order) models of a plant, which are commonly used in practice, instead of very detailed models of higher order
6. Ignorance of the structure and the model order at high frequencies
7. Controller order reduction issues and implementation inaccuracies.

The above sources of model uncertainty may be grouped into three main categories.

1.Parametric uncertainty representation

In this case the structure of model and its order is known, but some of the parameters are uncertain and vary in a subset of the parameter space.

2.Neglected and unmodeled dynamics uncertainty

In this case the model is in error because of missing dynamics, most likely due to a lack of understanding of the physical process.

3.Non-parametric uncertainty

In this case uncertainty represents several sources of parametric and/or unmodeled dynamics uncertainty combined into a single lumped perturbation of prespecified structure. Here, nothing is known about the exact nature of the uncertainties, except that they are bounded.

8.6.1 Parametric uncertainty representation

Parameter uncertainty means that parameter perturbation of controlled object is used to express uncertainty. Such uncertainty does not generally change system structure. In practical engineering, all kinds of parameters such as friction coefficient, vector, moment of inertia, network parameter is used to measure the change caused by error or component aging. Such change can be depicted by parameter perturbation.

If a system could be depicted by state space model, parameter uncertainty could be expressed as follows:

$$\left.\begin{aligned} \dot{x} &= f(x,\quad \theta) + g(x,\quad \theta)u \\ y &= h(x,\quad \theta) + d(x,\quad \theta)u \end{aligned}\right\} \tag{8.33}$$

Here, $x \in R^n$, $u \in R^m$, $y \in R^p$ are respectively state, control and input; f, g, h, d are the mapping function of suitable dimension; $\theta = [\theta_1,\quad \theta_2,\quad \cdots,\quad \theta_s]^T$ is unknown parameter vector, θ_i demonstrates the parameter of uncertainty factors caused by error or unknown perturbation.

For linear system, equation(8.33) can be written into the following form:

$$\left.\begin{aligned} \dot{x} &= A(\theta) + B(\theta)u \\ y &= C(\theta) + D(\theta)u \end{aligned}\right\} \tag{8.34}$$

Here, $A(\theta) \in R^{n\times n}$, $B(\theta) \in R^{n\times m}$, $C(\theta) \in R^{p\times n}$ and $D(\theta) \in R^{p\times m}$ are matrix function of unknown parameter θ. In example8.4, if perturbation parameters are marked as $\theta_1 = \Delta M$, $\theta_2 = \Delta\mu$, motion equation of system can be expressed as

$$\frac{d\upsilon}{dt} = -\frac{\mu_0 + \Delta\mu}{M_0 + \Delta M}\upsilon + \frac{1}{M_0 + \Delta M} f$$

Namely,

$$\dot{\upsilon} = A(\theta)\upsilon + B(\theta)f$$

Here, $\theta = [\theta_1, \quad \theta_2]^T$, $A(\theta) = -\dfrac{\mu_0 + \theta_2}{M_0 + \theta_1}$, $B(\theta) = \dfrac{1}{M_0 + \theta_1}$.

From above example, it's known that the structure of $A(\theta)$ and $B(\theta)$ is given and includes a lot of given parameters. Furthermore, when $\theta = 0$, matrices A_0 and B_0 confirm the nominal model of the system. Hence, generally, $A(\theta)$ and $B(\theta)$ are expressed into the additional form of nominal value and perturbation value, namely

$$A(\theta) = A_0 + \Delta A(\theta), \qquad B(\theta) = B_0 + \Delta B(\theta) \tag{8.35}$$

If possible, the given part of matrices $\Delta A(\theta)$ and $\Delta B(\theta)$ may be separated to be expressed into the following:

$$\Delta A(\theta) = E_a \sum(\theta) F_a, \quad \Delta B(\theta) = E_b \sum(\theta) F_b \tag{8.36}$$

In fact, the separation form is not unique. For above example8.4, system model can be expressed as

$$A_0 = A(0) = -\frac{\mu_0}{M_0}, \quad \Delta A(\theta) = A(\theta) - A_0 = -\frac{\mu_0 + \theta_2}{M_0 + \theta_1} + \frac{\mu_0}{M_0} = -\frac{\mu_0}{M_0}\left(\frac{-\theta_1/M_0 + \theta_2/\mu_0}{1 + \theta_1/M_0}\right)$$

Redefine unknown parameters as follows

$$\theta_1^{'} = -\frac{\theta_1/M_0}{1 + \theta_1/M_0}, \qquad \theta_2^{'} = \frac{\theta_2/\mu_0}{1 + \theta_1/M_0}$$

Then

$$\Delta A(\theta) = E_a \sum(\theta^{'}) F_a$$

Here, $E_a = [1 \quad 1]$, $\sum(\theta^{'}) = [\theta_1^{'} \quad \theta_2^{'}]^T$, $F_a = -\dfrac{\mu_0}{M_0}$.

Similarly, $\Delta B(\theta)$ may be expressed as follows

$$B_0 = B(0) = \frac{1}{M_0}, \quad \Delta B(\theta) = \frac{1}{M_0 + \theta_1} - \frac{1}{M_0} = \frac{1}{M_0}\left(\frac{-\theta_1/M_0}{M_0 + \theta_1}\right) = E_b \sum(\theta^{'}) F_b$$

Here, $E_b = [1 \quad 0]$, $F_b = \dfrac{1}{M_0}$.

Then, system model in equation(8.34) may be expressed as

$$\left.\begin{aligned} \dot{x} &= \left[A_0 + E_a \sum(\theta^{'}) F_a\right] + \left[B_0 + E_b \sum(\theta^{'}) F_b\right] u \\ y &= \left[C_0 + E_c \sum(\theta^{'}) F_c\right] + \left[D_0 + E_d \sum(\theta^{'}) F_d\right] u \end{aligned}\right\} \tag{8.37}$$

For nonlinear system, system model may be expressed as

$$\left.\begin{aligned} \dot{x} &= \left[f(x) + E_a \sum(\theta^{'}) F_a(x)\right] + \left[g(x) + E_b \sum(\theta^{'}) F_b(x)\right] u \\ y &= \left[h(x) + E_c \sum(\theta^{'}) F_c(x)\right] + \left[d(x) + E_d \sum(\theta^{'}) F_d(x)\right] u \end{aligned}\right\} \tag{8.38}$$

Hence, parametric uncertainty can be quantified by assuming that each uncertain parameter α is bounded within some region $[\alpha_{\min}, \quad \alpha_{\max}]$. In other words, there are parameter sets of the form:

$$\alpha_p = \alpha_m(1 + r_\alpha \Delta) \tag{8.39}$$

Here, α_m is the mean parameter value, $r_\alpha = (\alpha_{\max} - \alpha_{\min})/(\alpha_{\max} + \alpha_{\min})$ is the relative parametric uncertainty and Δ is any scalar satisfying $|\Delta| \le 1$.

Example8.5 Assume that the set of possible control component is given by

$$G_p(s) = \beta_p G_0(s), \quad \beta_{\min} \le \beta_p \le \beta_{\max}$$

Here β_p is an uncertain gain and $G_0(s)$ is a transfer function without uncertainty. Please try to determine the uncertainty description.

Solution

In terms of equation(8.39), write β_p as follows

$$\beta_p = \beta_m(1 + r_\beta \Delta), \quad |\Delta| \le 1$$

Here,

$$\beta_m = \frac{\beta_{\min} + \beta_{\max}}{2} \quad \text{and} \quad r_\beta = \frac{\beta_{\max} - \beta_{\min}}{\beta_{\max} + \beta_{\min}}$$

Obviously, β_m and r_β are the average gain and the relative magnitude of the gain uncertainty, respectively. Therefore, model set can be written as

$$G_p(s) = \beta_m G_0(s)[1 + r_\beta \Delta], \quad \|\Delta\| \le 1$$

The above expression for $G_p(s)$ is the uncertainty description of the set in this example8.5.

Example8.6 Assume that the set of possible control component is given by

$$G_p(s) = \frac{1}{\tau_p s + 1} G_0(s), \quad \tau_{\min} \le \tau_p \le \tau_{\max}$$

Please try to determine the uncertainty description of example8.6.

Solution

Similarly, write τ_p as follows

$$\tau_p = \tau_m(1 + r_\tau \Delta), \quad |\Delta| \le 1$$

Here,

$$\tau_m = \frac{\tau_{\min} + \tau_{\max}}{2} \quad \text{and} \quad r_\tau = \frac{\tau_{\max} - \tau_{\min}}{\tau_{\max} + \tau_{\min}}$$

The model set in example8.6 can be written as

$$G_p(s) = \frac{G_0(s)}{1 + \tau_m s + r_\tau \tau_m s \Delta} = \frac{G_0(s)}{1 + \tau_m s}[1 + W_{im}(s)\Delta]^{-1} = G(s)[1 + W_{im}(s)\Delta]^{-1}$$

Here,

$$G(s)=\frac{G_0(s)}{1+r_m s} \text{ and } W_{im}(s)=\frac{r_\tau \tau_m s}{1+\tau_m s}$$

Above expression for $G_p(s)$ is the inverse multiplicative form of the uncertainty set in example8.6.

Although parametric uncertainty is easily represented in some simple cases, it is avoided in most cases because of the following reasons:

1. It requires large efforts to model parametric uncertainty, particularly in the case of systems with a large number of uncertain parameters.
2. In many cases, the assumptions about the model and the parameters may be inexact. However, the description of a family of system through parametric uncertainty is very detailed and accurate.
3. In order to model uncertain systems through parametric uncertainty, the exact model structure is indispensable. Unmodeled dynamics can not then be incorporated in this description.

8.6.2 Non-parametric uncertainty representation

When uncertainty effect can not be expressed by parameter perturbation, it can be depicted by unknown perturbation function or unknown dynamic equation. This type of uncertainty is called to non-parametric uncertainty. For example, the nonlinear system uncertainty on state function perturbation may be expressed as

$$\left.\begin{aligned}\dot{x}&=[f(x)+\Delta f(x)]+[g(x)+\Delta g(x)]u\\ y&=[h(x)+\Delta h(x)]+[d(x)+\Delta d(x)]u\end{aligned}\right\} \tag{8.40}$$

Here, f, g, h, d are the given function vector or function matrix, $\Delta f(x)$, $\Delta g(x)$, $\Delta h(x)$ and $\Delta d(x)$ are unknown function vector or function matrix.

If nominal system is linear system, its non-parametric uncertainty model can be expressed as

$$\left.\begin{aligned}\dot{x}&=[A+\Delta A(x)]+[B+\Delta B(x)]u\\ y&=[C+\Delta C(x)]+[D+\Delta D(x)]u\end{aligned}\right\} \tag{8.41}$$

In equation(8.40), matrices ΔA, ΔB, ΔC, ΔD are unknown matrices.

Similarly, when we build a above mathematical model of control system, the known part in perturbation function should be separated to reduce the conservation when a robust control system is designed. For example, system function can be separated as

$$\Delta f(x)=E(x)\sum(x) \tag{8.42}$$

Here, $E(x)$ is known matrix, $\sum(x)$ is unknown function vector. Or ΔA is expressed as

$$\Delta A(x) = E\sum F \tag{8.43}$$

Here, E, F are known matrices, and $\sum$ is unknown matrix.

Uncertainty which is depicted by static function perturbation can not change the dimension of system, that's to say, it does not add the number of system state variable. However, in practice, some factors are dynamic, and then they have to be depicted by independent variables.

For linear system, frequency characteristic or transfer function can be employed to describe system. For example, transfer function with uncertainty can be expressed as follows:

$$G(s) = G_0(s) + \Delta G(s) \tag{8.44}$$

In equation(8.43), function $G_0(s)$ is the transfer function of nominal system, and $\Delta G(s)$ is the transfer function of unknown dynamic system. If $\Delta G(s)$ is bounded and if its upper limitation is given, namely if rational function makes the following inequality be found

$$\sigma_{\max}(\Delta G(j\omega)) \le \|r(j\omega)\|, \quad \forall \omega \in R \tag{8.45}$$

then frequency characteristic curve of the transfer function in equation(8.43) will be drawn in amplitude-frequency characteristic plane. Namely, equation(8.44) and equation(8.45) describe same system.

For nonlinear system, control system with uncertainty can be described as

$$\begin{cases} \dot{x} = f(x, \quad \eta) + g(x, \quad \eta)u \\ \varepsilon = Q(x, \quad \varepsilon) \\ \eta = P(\varepsilon) \end{cases} \tag{8.46}$$

In equation(8.46), ε is utilized to describe the unknown state of uncertainty, Q and P are unknown vector function.

This description steps of non-parametric uncertainty model are as follows:

1.Choose a nominal model $G(s)$. A nominal model can be selected to be either a low-order, delay-free or a model of mean parameter values or , finally, the central plant obtained from the Nyquist plots corresponding to all of the plants of the given set P.

2.In the case of additive uncertainty, find the smallest radius $r(j\omega)$, which includes all possible plants.

$$r_a(j\omega) = \max_{G_p \in P} |G_P(j\omega) - G(j\omega)| \tag{8.47}$$

In most cases we look for a rational transfer function weight $W_a(s)$ for additive uncertainty,. This weight must be chosen such that

$$|W_a(j\omega)| \ge r_a(j\omega), \quad \forall \omega \tag{8.48}$$

and is usually selected to be of low order to simplify the design of controllers.

3.In the case of multiplicative uncertainty which is the preferred uncertainty form, find the smallest radius $r_m(j\omega)$, which includes all possible executing components.

$$r(j\omega) = \max_{G_p \in P_m} \left| \frac{G_P(j\omega) - G(j\omega)}{G(j\omega)} \right| \tag{8.49}$$

For a chosen rational weight $W_m(s)$, there must be

$$|W_m(j\omega)| \geq r_m(j\omega), \quad \forall \omega \tag{8.50}$$

8.7 Robust Model of Linear Uncertainty System

For linear system, the transfer function $G_0(s)$ of nominal system and unknown transfer function error $\Delta G(s)$ and its boundary function $r(s)$ are employed to depict uncertain system.

In fact, both nominal system model and uncertain model structure are not unique. In spite of which structure is utilized to express given system, nominal model $G_0(s)$ and certain boundary function $r(s)$ need to be constructed when the linear system model with uncertainty is built.

8.7.1 Types of uncertainty system model

Expression forms of several common uncertainty system set are as follows:

(1) Multiplication uncertainty(multiplication perturbation). Its system model is shown in figure8.10.

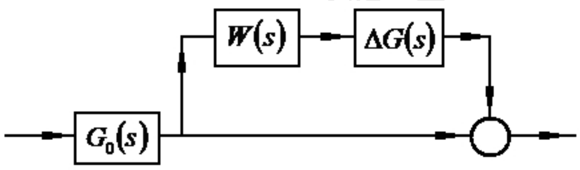


Figure8.10 Multiplication perturbation

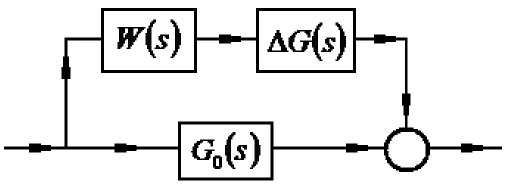


Figure8.11 Addition perturbation

$$G(s) = [1 + \Delta G(s) W(s)] G_0(s), \qquad \|\Delta G(s)\|_\infty < 1 \tag{8.51}$$

In equation(8.33), $G_0(s)$ is certainty model(nominal model), $\Delta G(s)$ is unknown perturbation function, and $W(s)$ shows perturbation boundary function of $\Delta G(s)$, which is also called to power function that is generally a stable transfer function.

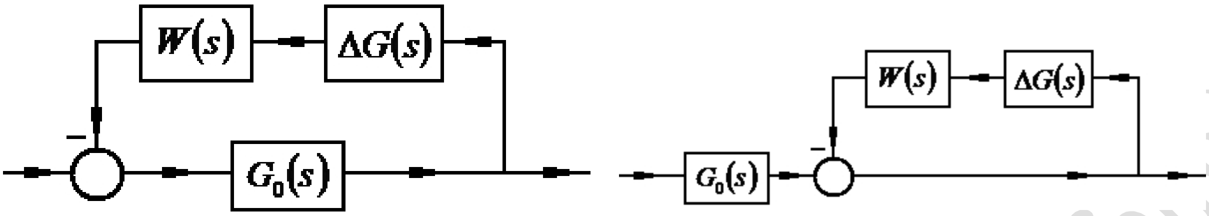
(2) Addition uncertainty(addition perturbation). Its system model is shown in figure8.11.

$$G(s) = G_0(s) + \Delta G(s) W(s), \qquad \|\Delta G(s)\|_\infty < 1 \tag{8.52}$$

(3) Feedback uncertainty(feedback perturbation). Its system model is shown in figure8.12.

Ⅰ type of system model:

$$G(s)=\frac{G_0(s)}{1+\Delta G(s)W(s)G_0(s)}, \qquad \|\Delta G(s)\|_\infty<1 \tag{8.53}$$



(a) Ⅰ type (b) Ⅱ type

Figure8.12 Feedback uncertainty model

Ⅱ type of system model:

$$G(s)=\frac{G_0(s)}{1+\Delta G(s)W(s)}, \qquad \|\Delta G(s)\|_\infty<1 \tag{8.54}$$

For model uncertainty of above types, their perturbation value can be measured by H_∞ norm.Hence, above model perturbation is called to norm boundary perturbation. In practice, what is important is to how to confirm the boundary of perturbation function using given information.

Example8.7 Assume that the change range of car mass is as follows:

$$|M-M_0|\le c, \quad c>0$$

Then, mass M could be expressed as

$$M=M_0+c\Delta, \qquad -1\le\Delta\le 1$$

Hence, transfer function can be expressed into the form of feedback perturbation:

$$G(s)=\frac{1}{Ms+\mu}=\frac{1}{M_0s+\mu+\Delta cs}=\frac{1}{1+\Delta W(s)}G_0(s)$$

Here,

$$G_0(s)=\frac{1}{M_0s+\mu}, \quad W(s)=\frac{cs}{M_0s+\mu}, \qquad |\Delta|\le 1$$

It is noted that above model is only the subset of equation(8.54) for coefficient Δ is not complex but real.

8.7.2 Model of perturbation boundary function

From above examples, it is known that confirming the boundary function of perturbation function is a very important component in spite of which structure is adopted. It's very difficult that a uniform algorithm which is used to confirm the boundary function of perturbation function is provided for readers. However, whatever methods are adopted to find boundary function of perturbation function, the conservatism of model should be possibly reduced to make boundary function including perturbation be closer to actual situation. Generally, weighting function $W(s)$ in the domain of medium-high frequency should not exceed the gain of perturbation. It is because accurate control is very difficult in a large range of perturbation frequency band. In frequency band, if given system characteristic is accurate and given, control performance can be improved when designed.

The following several design methods of perturbation boundary function $W(s)$ are as follows:

1. System identification method

Firstly, the difference $|G(j\omega)-G_0(j\omega)|$ between the frequency response $G_0(j\omega)$ of actual system and the frequency response $G_0(j\omega)$ of nominal system is calculated out and drawn in complex plane. Then weighting function $W(s)$ is ensured to make it cover the difference $|G(j\omega)-G_0(j\omega)|$.

2. Approximation method

Approximation method in fact is a kind of imminent method that utilizes low-order system $G_0(s)$ approximately approaches high-order system $G(s)$. Namely, the following expressions should be calculated out.

$$|G(j\omega)-G_0(j\omega)| \quad \text{or} \quad \left|1-\frac{G(j\omega)}{G_0(j\omega)}\right| \tag{8.55}$$

Secondly, weighting function $W(s)$ is acquired in the Bode plane to meet the following inequality:

$$|G(j\omega)-G_0(j\omega)| \le |W(j\omega)| \quad \text{(additional perturbation)} \tag{8.56a}$$

Or,

$$\left|1-\frac{G(j\omega)}{G_0(j\omega)}\right| \le |W(j\omega)| \quad \text{(multiplicative perturbation)} \tag{8.56b}$$

Namely, the left of inequality (8.56) is drawn out, then weighting function $W(s)$ is made to cover it.

8.8 Robust Stability in the H_∞ Context

In previous section, author has discussed how to represent model uncertainty in a mathematical context. In this section author will derive conditions under which a control system remain stable for all perturbations in an uncertainty set.

8.8.1 Robust stability with a multiplicative uncertainty

In figure8.13, a feedback system with a plant $H(s)$, a controller $K(s)$, and multiplicative uncertainty is represented. If there is a multiplicative uncertainty of magnitude $|W(j\omega)|$ it is determined whether the uncertain feedback system is stable.

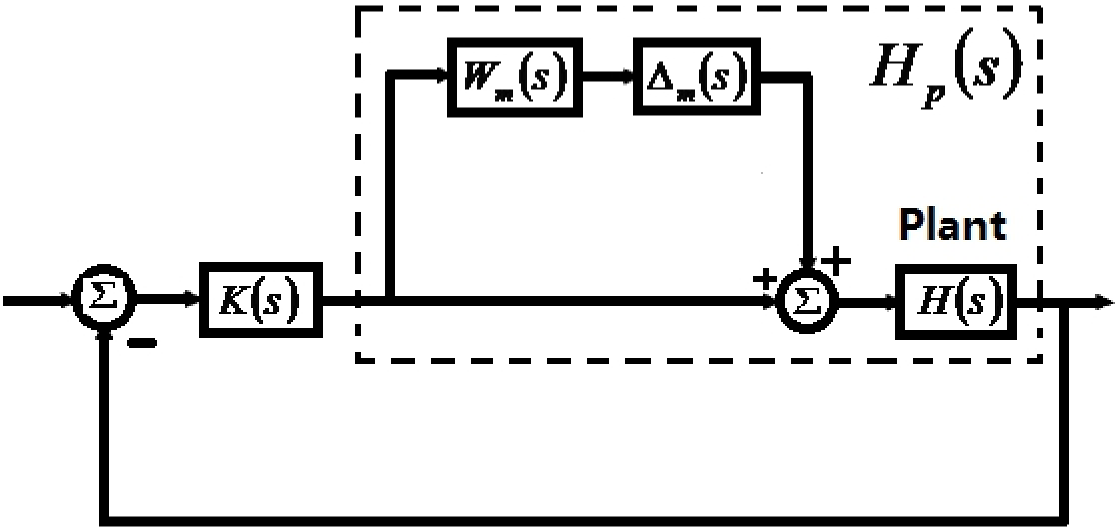


Figure8.13 Closed-loop feedback system with multiplicative uncertainty

With the uncertainty present, the open-loop transfer function of feedback system in figure8.13 is given by

$$G_p(s) = H_p(s)K(s) = H(s)K(s)[1 + W_m(s)\Delta_m(s)] = G(s) + W_m(s)G(s)\Delta_m(s) \qquad (8.57)$$

Here, $|\Delta_m(s)| \le 1$.

Assume that, by design, the stability of the nominal closed-loop control system in figure8.13 can be guaranteed. Furthermore, for simplicity, also assume that the transfer function of the open-loop control system is stable. To test for the stability of the closed-loop control system, Nyquist stability criterion needs to be utilized. Then it is known that Nyquist stability, which is equivalent to the stability of all system for all $G_p(s)$, is also equivalent to the fact that open-loop transfer function $G_p(s)$ should not encircle the point $-1 + j0$, for all $G_p(s)$.

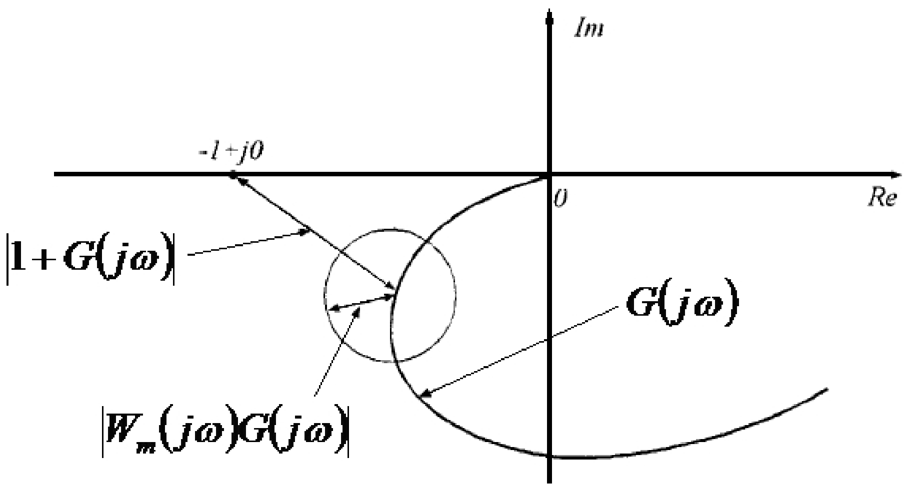


Figure8.14 Graphical derivation of robust stability condition through Nyquist plot

Now let's consider a typical plot of $G_p(s)$ as shown in figure8.14. The distance from the point $-1+j0$ to the center of disk, which represents $G_p(s)$, is $|1+G(s)|$. Furthermore, the radius of this disk is $|W_m(s)G(s)|$. To avoid encirclement of point $-1+j0$, none of disks should cover the critical point. Through inspecting the curve in figure8.14, it is concluded that the encirclement is avoided if and only if

$$|W_m(s)G(s)| < |1+G(s)|, \quad \forall_\omega \tag{8.58}$$

or equivalently if and only if

$$\left|\frac{W_m(s)G(s)}{1+G(s)}\right| < 1, \quad \forall_\omega \tag{8.59}$$

Definition of sensitivity

The sensitivity function of system designated by $S(s)$ and the complementary sensitivity function marked by $T(s)$ are defined as follows:

$$S(s) = [1+G(s)]^{-1} = [1+H(s)K(s)]^{-1} \tag{8.60a}$$

$$T(s) = H(s)K(s)[1+G(s)]^{-1} \tag{8.60b}$$

The sensitivity functions $S(s)$ and $T(s)$ satisfy the relation: $S(s)+T(s)=1$.

Using equation(8.60), one can conclude that the encirclement is avoided (equivalently the robust stability condition is satisfied) if and only if

$$|W_m(s)T(s)| < 1, \quad \forall\omega \tag{8.61}$$

The robust stability under multiplicative perturbation is assumed if and only if

$$\|W_m(s)T(s)\|_\infty < 1 \tag{8.62}$$

It is worth noting that the robust stability condition(8.62) for the case of the multiplicative uncertainty gives an upper bound on the complementary sensitivity function. In other words, to guarantee robust stability in the case of multiplicative uncertainty one has to make $T(s)$ small at frequencies where the uncertainty weight exceeds 1 in magnitude.

Condition(8.62) is necessary and sufficient provided that, at each frequency, all perturbations satisfying $|\Delta_m(j\omega)| < 1$ are possible for the feedback system studied. If this is not the case, this condition is only sufficient.

An alternative, rather algebraic, way of obtaining the robust stability condition(8.62) is the following.

Since function $G_p(s)$ is assumed to be stable and the nominal closed-loop system is stable by design, then nominal open-loop transfer function does not encircle the critical point $-1+j0$. Consequently, since the family of uncertain plants is norm bounded, it then follows that, if for some $G_{p1}(s)$ in the uncertain family, we have encirclement of point $-1+j0$, then there must be another function $G_{p2}(s)$ in the uncertain family, which passes through point $-1+j0$ at some frequency. Therefore, to guarantee robust stability, the following condition must hold:

$$\left|1+G_p(s)\right| \neq 0, \quad \forall G_p, \quad \forall \omega \tag{8.63}$$

Hence, robust stability is guaranteed if and only if

$$\left|1+G(s)+W_m(s)G(s)\Delta_m(s)\right| > 0, \quad \forall\left|\Delta_m(s)\right| \leq 1, \quad \forall \omega \tag{8.64}$$

This last condition is most easily violated at each frequency when $\Delta_m(j\omega)$ has magnitude 1 and the phase is such that the terms $1+G(s)$ and $W_m(s)G(s)\Delta_m(s)$ have opposite signs. Thus, robust stability is guaranteed if and only if

$$\left|1+G(s)\right| - \left|W_m(s)G(s)\right| > 0, \quad \forall \omega \tag{8.65}$$

Then condition(8.62) follows easily.

Here, an example is provided to check robust stability using multiplicative perturbation.

Example8.8 Consider the uncertain feedback control system in figure8.13. Assume that the uncertain plant transfer function is given by the following expression:

$$H_p(s) = H(s)[1+W_m(s)\Delta_m(s)]$$

Where

$$H(s) = \frac{1}{s-1} \quad \text{and} \quad W_m = \frac{2}{s+10}$$

While the controller $K(s)$ is a constant gain controller of the form $K(s)=10$. Please determine and judge whether the closed-loop system is robustly stable.

Solution

For this case the complementary sensitivity function $T(s)$ is given by

$$G(s) = H(s)K(s) = \frac{10}{s-1}, \quad S(s) = [1+G(s)]^{-1} = \frac{s-1}{s+9}$$

$$T(s) = 1-S(s) = \frac{10}{s+9}, \qquad W_m(s)T(s) = \frac{2}{s+10}\frac{10}{s+9} = \frac{20}{s^2+19s+90}$$

$$\lim_{\omega\to\infty}\left\|W_m(s)T(s)\right\| = \lim_{\omega\to\infty}\frac{20}{\sqrt{\left(90-\omega^2\right)^2+(19\omega)^2}} = 0 < 1$$

In terms of given condition(8.62), we can know that the closed-loop system is robustly stable.

8.8.2 Robust stability with an inverse multiplicative uncertainty

In reality, the closed-loop system with inverse multiplicative uncertainty is possible. Hence in this section, the robust stability condition will be derived for the case of feedback systems with inverse multiplicative uncertainty.

Now the system block diagram is shown in figure8.15, with plant transfer function

$H(s)$, a controller $K(s)$, and an inverse multiplicative uncertainty of magnitude $W_m(s)$.

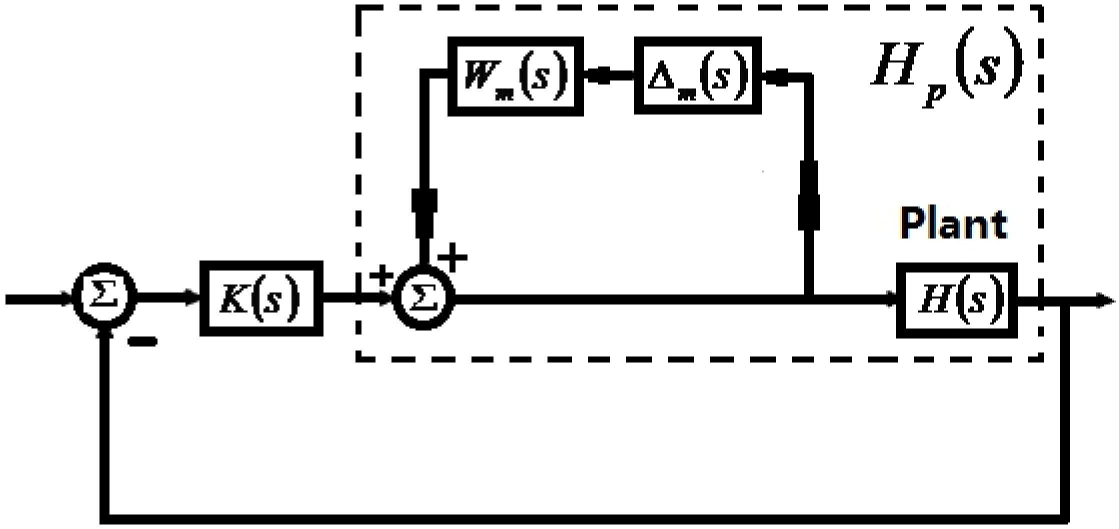


Figure8.15 Closed-loop control system with inverse multiplicative uncertainty

Here,

$$H_p(s) = H(s)[1 + W_{im}(s)\Delta_{im}(s)]^{-1} \tag{8.66}$$

Now, suppose that the open-loop transfer function $G_p(s)$ is stable and that the nominal closed-loop system is also stable. As mentioned above, and since $G_p(s)$ belongs to a norm-bounded set, one conclude that robust stability is guaranteed if and only if one of the following four equivalent inequalities holds:

$$|1 + G_p(s)| > 0, \quad \forall G_p(s), \quad \forall \omega \tag{8.67a}$$

$$\left|1 + G(s)[1 + W_{im}(s)\Delta_{im}(s)]^{-1}\right| > 0, \quad \forall |\Delta_{im}(j\omega)| \le 1, \quad \forall \omega \tag{8.67b}$$

$$\left|\frac{1 + G(s) + W_{im}(s)\Delta_{im}(s)}{1 + W_{im}(s)\Delta_{im}(s)}\right| > 0, \quad \forall |\Delta_{im}(j\omega)| \le 1, \quad \forall \omega \tag{8.67c}$$

$$|1 + G(s) + W_{im}(s)\Delta_{im}(s)| > 0, \quad \forall |\Delta_{im}(j\omega)| \le 1, \quad \forall \omega \tag{8.67d}$$

The last condition is most easily violated at each frequency when $\Delta_{im}(j\omega)$ has magnitude 1 and the phase is such that the terms $1 + G(s)$ and $W_{im}(s)\Delta_{im}(s)$ have opposite signs. Thus, the robust stability is guaranteed if and only if

$$|1 + G(s)| - |W_m(s)| > 0, \quad \forall \omega \tag{8.68}$$

Taking into account the definitions of the sensitivity function $S(s)$ and of the H_∞ norm, we finally obtain that robust stability with inverse multiplicative uncertainty is guaranteed if and only if

$$\|W_{im}(s)S(s)\|_\infty < 1 \tag{8.69}$$

Condition(8.69) indicates that in order to guarantee robust stability, in the case of the inverse multiplicative perturbation, one has to make $S(s)$ small at frequencies where uncertainty weight exceeds 1 in magnitude.

Example8.9 Consider the uncertain feedback control system in figure8.15. Assume that the uncertain plant transfer function is given by the following expression:

$$H_p(s) = H(s)[1 + W_{im}(s)\Delta_{im}(s)]^{-1}$$

Where

$$H(s) = \frac{1}{s-1} \quad \text{and} \quad W_{im} = \frac{s+2}{3s+0.7}$$

While the PI controller $K(s)$ is a constant gain controller of the form

$$K(s) = 1 + \frac{2}{s}$$

Please determine and judge whether the closed-loop system is robustly stable.

Solution

For this case the sensitivity function $S(s)$ is given by

$$G(s) = H(s)K(s) = \frac{1}{s-1}\frac{s+2}{s} = \frac{s+2}{s^2-s}, \qquad S(s) = [1+G(s)]^{-1} = \frac{s^2-s}{s^2+2}$$

$$W_{im}(s)S(s) = \frac{s+2}{3s+0.7}\frac{s^2-s}{s^2+2} = \frac{s^3+s^2-2s}{3s^3+0.7s^2+6s+1.4}$$

$$\lim_{\omega\to\infty}\|W_{im}(s)S(s)\| = \frac{1}{3} < 1$$

In terms of equation(8.69), it is known that the closed-loop system is robustly stable.

For other cases of uncertainty descriptions such as the additive or the division uncertainty, one can easily obtain robust stability conditions analogous to the given conditions(8.62) and (8.69). Table8.3 summarizes the robust stability conditions for several commonly used uncertainty models.

Table8.3 Robust stability conditions for several cases

Uncertainty description	Robust stability condition
Additive uncertainty $G(s)+W_a(s)\Delta_a(s)$	$\|W_a(s)K(s)S(s)\|_\infty < 1$
Multiplicative uncertainty $G(s)[1+W_m(s)\Delta_m(s)]$	$\|W_m(s)T(s)\|_\infty < 1$
Inverse multiplicative uncertainty $G(s)[1+W_{im}(s)\Delta_{im}(s)]^{-1}$	$\|W_{im}(s)S(s)\|_\infty < 1$
Division uncertainty $G(s)[1+W_d(s)G(s)\Delta_d(s)]^{-1}$	$\|W_d(s)G(s)S(s)\|_\infty < 1$

Perturbations always affect the performance of plant in reality, therefore, it’s necessary for us to study the performance of robust control system. Here what we narrates robust performance is internal stability and performance which should hold for all plants in a family P. Before dealing with robust performance, nominal performance and its relation to the sensitivity function should be firstly reviewed in brief. In the following, robust performance will be explored for readers.

8.9 Robust Performance in the H_∞ Context

8.9.1 Nominal performance

The system block diagram in figure8.16 will be considered. Here, $H(s)$ is the plant transfer function, $K(s)$ is the controller transfer function, $r(t)$ or $R(s)$ is the reference input(command, set-point), $d(t)$ or $D(s)$ is the disturbance(process noise), $n(t)$ or $N(s)$ is the measurement noise, $y_m(t)$ or $Y_m(s)$ is the measured output, and $u(t)$ or $U(s)$ is control signal(actuator signal).

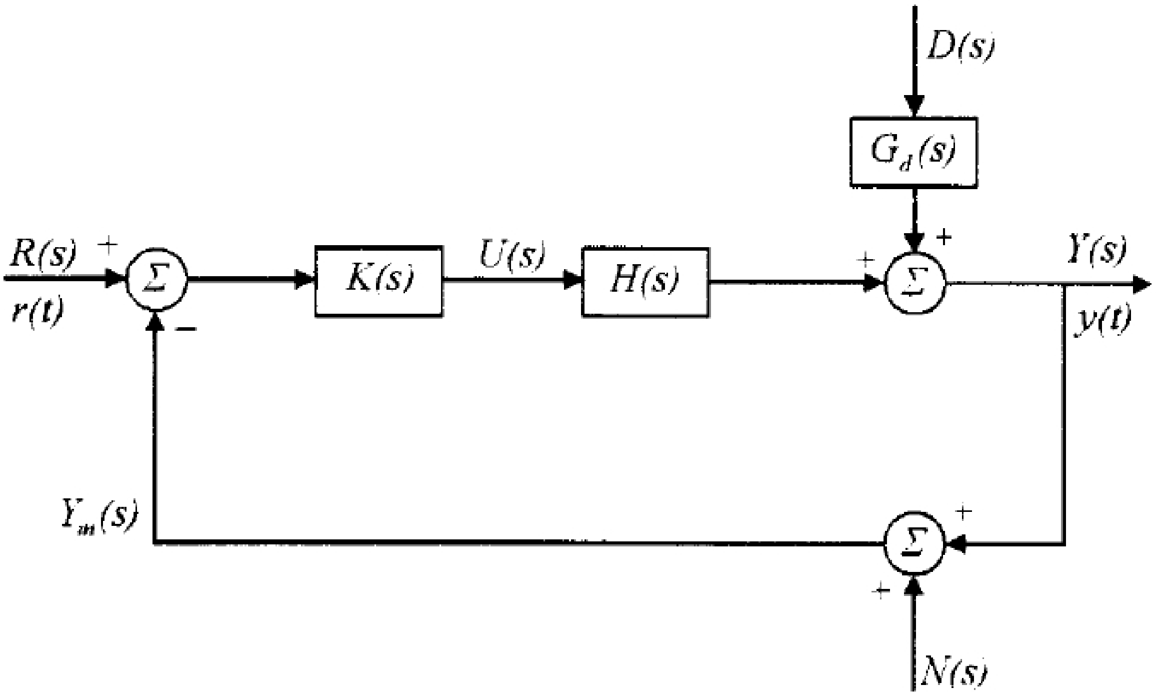


Figure8.16 Block diagram of control system with disturbance and noise

According to figure8.16, the control error may be expressed as

$$e(t)=r(t)-y(t), \quad \text{or,} \quad E(s)=R(s)-Y(s) \tag{8.70}$$

Namely,

$$E(s)=[1+K(s)H(s)]^{-1}[R(s)+G_d(s)D(s)]-K(s)H(s)[1+K(s)H(s)]^{-1}N(s) \tag{8.71}$$

Or, in terms of the sensitivity and the complementary sensitivity functions,

$$E(s)=S(s)R(s)+S(s)G_d(s)D(s)-T(s)N(s) \tag{8.72}$$

If perfect control is gained from figure8.16, control error should be zero, namely, $e(t)=r(t)-y(t)=0$. That's to say, good disturbance should be rejected, command tracking and measurement noise on the plant output should be reduced. This means that, for disturbance rejection and command tracking, the sensitivity function $S(s)$ must be chosen to be small in magnitude,whereas for zero noise transmission the same function must have a large magnitude, close to 1 in the case of that $T(s)$ is small in magnitude. This illustrates the fundamental nature of feedback design, which always involves a trade-off among conflicting control objectives. Moreover, it illustrates that the sensitivity function $S(s)$ is very good indicator of closed-loop performance. In particular, when considering $S(s)$ as such an indicator, main advantage stems from the fact that it is sufficient to consider just its magnitude and not worry about its phase.

Some common specifications in terms of sensitivity function $S(s)$ are listed as follows:

1. Maximum tracking error at pre-specified frequencies
2. Minimum steady-state tracking error A
3. Maximum peak magnitude M of sensitivity function $S(s)$
4. Minimum bandwidth ω_b^*

Performance specifications of the above type in Table8.3 can usually be incorporated in an upper bound, $1/W_p(s)$, on the magnitude of the sensitivity function, where $W_p(s)$ is a weight function chosen by the designer. The subscript P stands for performance, since, as already mentioned, the sensitivity function is used as a performance indicator. Then, the performance requirement is guaranteed if and only if one of the following three equivalent inequalities is found:

$$|S(j\omega)| < 1/|W_p(j\omega)|, \quad \forall\omega \tag{8.73a}$$

$$|W_p(j\omega)S(j\omega)| < 1, \quad \forall\omega \tag{8.73b}$$

$$\|W_p(j\omega)S(j\omega)\|_\infty < 1 \tag{8.73c}$$

A typical weight function is as follows:

$$W_p(s) = \frac{s/M + \omega_b^*}{s + \omega_b^* A} \tag{8.74}$$

The following basic conclusions can be easily obtained from equation(8.74).

1.When $s \to 0, \quad S(s) \to A$

2. When $s \to \infty, \quad S(s) \to M$

3. The asymptotic curve of the Bode plot of the magnitude of sensitivity function $S(s)$ cross 0 dB, at the frequency ω_b^*, which is the bandwidth requirement.

Now, figure8.17 is considered to acquire the equivalent condition of nominal performance. Taking into account the definition of sensitivity function, one can acquire from equation(8.73) that nominal performance is equivalent to

$$|W_p(j\omega)| < |1 + G(j\omega)|, \quad \forall\omega \tag{8.75}$$

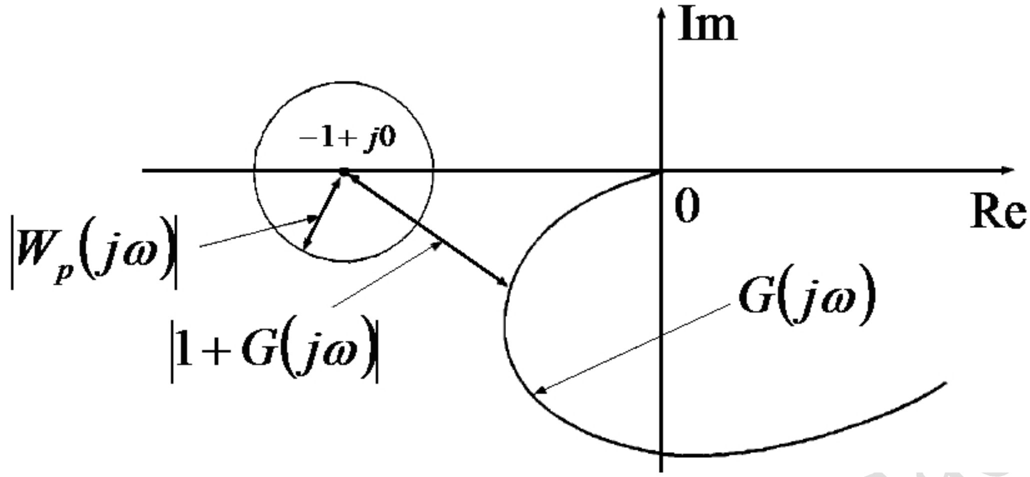


Figure8.17 Nominal performance in the Nyquist plot

At each frequency, the term $|1+G(s)|$ is the distance of $G(s)$ from the critical point $-1+j0$ in the Nyquist plot. Hence, for nominal performance, $G(j\omega)$ must be at least at a distance of weight function $|W_p(j\omega)|$ from the critical point. In other words, for nominal performance, $G(j\omega)$ must stay outside a disk of radius $|W_p(j\omega)|$, centered at $-1+j0$. This graphical interpretation of nominal performance is depicted in figure8.17.

Example8.10 Consider the uncertain feedback control system in figure8.16 with

$$H(s)=\frac{1}{s-1} \quad \text{and} \quad K(s)=10$$

The design performance specifications for the closed-loop control system be the following:

1.Steady-state tracking error $A=0.2$

2.Maximum peak magnitude $M=2$ of sensitivity function $S(s)$

3.Minimum bandwidth $\omega_b^*=0.5rad/\sec$

Please judge whether the performance specifications for the closed-loop control system meet the nominal performance requirement.

Solution

In terms of equation(8.74), the weighting function can be acquired as

$$W_p(s)=\frac{s/M+\omega_b^*}{s+\omega_b^* A}=\frac{s/2+1/2}{s+0.5\times0.2}=\frac{s+1}{2s+0.2}$$

$$G(s)=H(s)K(s)=\frac{1}{s-1}\times10=\frac{10}{s-1}, \qquad S(s)=[1+G(s)]^{-1}=\frac{s-1}{s+9}$$

$$1+G(s)=\frac{s+9}{s-1}$$

$$\|1+G(j\omega)\|_\infty=\lim_{\omega\to\infty}\frac{\sqrt{9^2+\omega^2}}{\sqrt{1^2+\omega^2}}=\lim_{\omega\to\infty}\sqrt{\frac{9^2+\omega^2}{1^2+\omega^2}}=1, \quad \|W_p(j\omega)\|_\infty=\lim_{\omega\to\infty}\frac{\sqrt{1^2+\omega^2}}{\sqrt{0.2^2+4\omega^2}}=0.5$$

Apparently, $\|1+G(j\omega)\|_\infty$ is greater than $\|W_p(j\omega)\|_\infty$, that's, the given condition in equation(8.75) can be satisfied. Therefore, the closed-loop system could meet the nominal performance requirement.

8.9.2 Robust performance

Clearly, for robust performance, it is sufficient to require that condition(8.75) is satisfied for all possible plants $G_p(s)$. In mathematical terms, robust performance is defined by one of the following two equivalent inequalities:

$$|W_p(j\omega)| < |1+G_p(j\omega)|, \quad \forall G_p, \quad \forall \omega \tag{8.76a}$$

$$|W_p(j\omega)S_p(j\omega)| < 1, \quad \forall S_p, \quad \forall \omega \tag{8.76b}$$

In order to explore the robust performance and to simplify the model, one needs to focus on multiplicative uncertainty case which is shown in figure8.18.

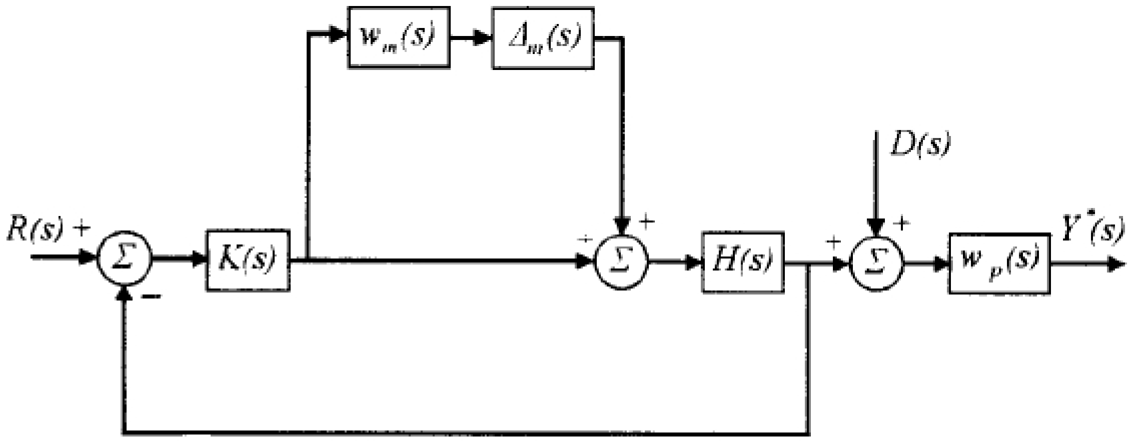


Figure8.18 Robust performance block diagram in multiplicative uncertainty case

Figure8.18 presents the block diagram for robust performance in the multiplicative uncertainty case. It is not difficult to understand that condition(8.76) corresponds to the requirement $|y^*/d| < 1, \quad \forall \Delta_m$. In this case the set of possible open-loop transfer function is provided by

$$G_p(s) = K(s)H_p(s) = G(s)[1+W_m(s)\Delta_m(s)] = G(s) + W_m(s)G(s)\Delta_m(s) \tag{8.77}$$

Now, consider the Nyquist plot in figure8.19. To guarantee robust performance, it is required that all possible open-loop transfer functions $G_p(j\omega)$ stay outside a disk of radius $|W_p(j\omega)|$ centered on the critical point $-1+j0$. It is evident that $G_p(j\omega)$, at each frequency, stay within a disk of radius $W_p(j\omega)G(j\omega)$, centered on $G(j\omega)$. Therefore, in figure8.19, the condition for robust performance is that these two disks must not overlap. The distance between the two centers of two disks is $|1+G(j\omega)|$. Consequently, the robust performance is guaranteed if and only if one of the following two equivalent inequalities is satisfied:

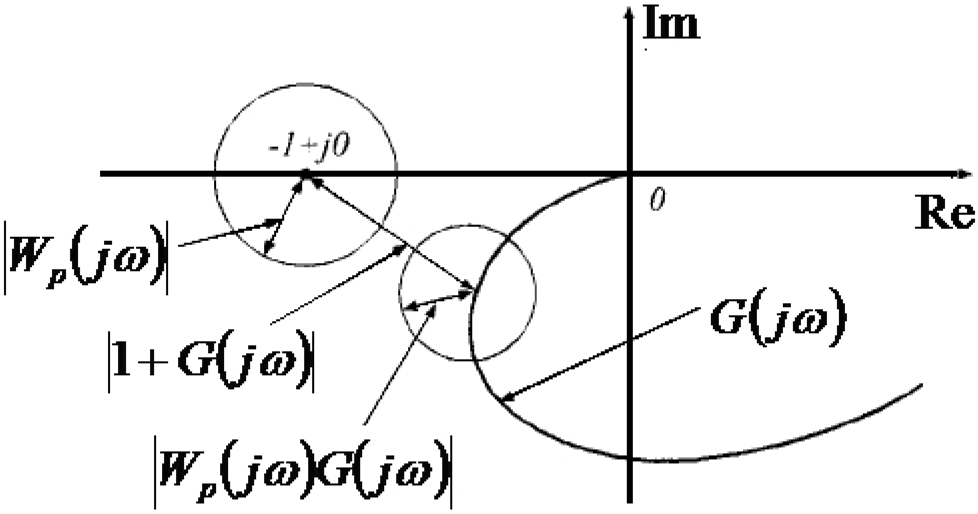


Figure8.19 Robust performance in Nyquist plot

$$|W_p(j\omega)| + |W_m(j\omega)G(j\omega)| < |1+G(j\omega)|, \quad \forall\omega \tag{8.78a}$$

$$\left|W_p(j\omega)[1+G(j\omega)]^{-1}\right| + \left|W_m(j\omega)G(j\omega)[1+G(j\omega)]^{-1}\right| < 1, \quad \forall\omega \tag{8.78b}$$

In terms of the definitions of the sensitivity and the complementary sensitivity functions, robust performance can be obtained if and only if

$$|W_p(j\omega)S(j\omega)| + |W_m(j\omega)T(j\omega)| < 1, \quad \forall\omega \tag{8.79}$$

Or in another form,

$$\left\| |W_p(j\omega)S(j\omega)| + |W_m(j\omega)T(j\omega)| \right\|_\infty < 1, \quad \forall\omega \tag{8.80}$$

Here, equation(8.80) is given from the definition of the H_∞-norm. Relation(8.80) is a necessary and sufficient condition for robust performance.

An alternative way of acquiring the robust performance condition(8.79) is as follows.

According to the relation(8.76a), robust performance is guaranteed if the maximum weighted sensitivity $W_p(s)S(s)$, at each frequency, is less than 1 in magnitude. That's to say, robust performance is guaranteed if and only if

$$\sup_{S_p} |W_p(j\omega)S_p(j\omega)| < 1, \quad \forall\omega \tag{8.81}$$

The perturbation sensitivity is

$$S_p(s) = [1+G_p(s)]^{-1} = \frac{1}{1+G(s)+W_m(s)G(s)\Delta(s)} \tag{8.82}$$

The worst case(maximum) is obtained in the case that $|\Delta_m(s)| = 1$ at each frequency, such that the signs of the terms $1+G(s)$ and $W_m(s)G(s)\Delta_m(s)$ are opposite. In mathematical terms, the following expression can be acquired:

$$\sup_{S_p}\left|W_p(j\omega)S_p(j\omega)\right| = \frac{\left|W_p(j\omega)\right|}{\left|1+G(j\omega)\right|-\left|W_p(j\omega)G(j\omega)\right|} = \frac{\left|W_p(j\omega)S(j\omega)\right|}{1-\left|W_p(j\omega)T(j\omega)\right|} \tag{8.83}$$

Combining relations(8.81) and (8.83) and considering the definition of the H_∞-norm, one can readily obtain the robust performance condition(8.80).

Condition(8.80) provides us with some useful bounds on the magnitude of $G(s)$. In particular, by observing that $1-\left|G(j\omega)\right| \le \left|1+G(j\omega)\right|$ and $\left|G(j\omega)\right|-1 \le \left|1+G(j\omega)\right|$, the robust performance condition(8.80) is satisfied if

$$\frac{1+\left|W_p(j\omega)\right|}{1-\left|W_m(j\omega)\right|} < \left|G(j\omega)\right|, \quad \forall\omega: \left|W_m(j\omega)\right| < 1 \tag{8.84a}$$

Or if

$$\left|G(j\omega)\right| < \frac{1-\left|W_p(j\omega)\right|}{1+\left|W_m(j\omega)\right|}, \quad \forall\omega: \left|W_p(j\omega)\right| < 1 \tag{8.84b}$$

It is proved that the robust performance condition is satisfied if

$$\frac{\left|W_p(j\omega)\right|-1}{1-\left|W_m(j\omega)\right|} < \left|G(j\omega)\right|, \quad \forall\omega: \left|W_m(j\omega)\right| < 1 \, and \quad \left|W_p(j\omega)\right| > 1 \tag{8.85a}$$

Or if

$$\left|G(j\omega)\right| < \frac{1-\left|W_p(j\omega)\right|}{\left|W_m(j\omega)\right|-1}, \quad \forall\omega: \left|W_p(j\omega)\right| < 1 \, and \quad \left|W_m(j\omega)\right| > 1 \tag{8.85b}$$

For SISO system, the term $\left|W_p(s)S(s)\right|+\left|W_m(s)T(s)\right|$ is the structured singular value $\mu(\omega)$. With this definition, robust performance is guaranteed if and only if

$$\left\|\mu(\omega)\right\|_\infty < 1 \tag{8.86}$$

On the structured singular value, readers may result in other control books.

Example8.11 Consider the uncertain feedback control system in figure8.16 with

$$H(s) = \frac{1}{s-1} \quad K(s) = 10 \qquad W_m = \frac{2}{s+10} \quad \text{and} \quad W_p = \frac{s+1}{2s+0.2}$$

Please determine whether the closed-loop control system satisfies the robust performance specifications.

Solution

From the previous examples, it is known that the given system is robustly stable.

$$G(s) = H(s)K(s) = \frac{1}{s-1}\times 10 = \frac{10}{s-1}, \quad S(s) = \left[1+G(s)\right]^{-1} = \frac{s-1}{s+9}, \quad T(s) = \frac{10}{s+9}$$

$$\left|W_p(s)S(s)\right|+\left|W_m(s)T(s)\right| = \left|\frac{s+1}{2s+0.2}\times\frac{s-1}{s+9}\right| + \left|\frac{2}{s+10}\times\frac{10}{s+9}\right| = \left|\frac{s^2-1}{2s^2+18.2s+1.8}\right| + \left|\frac{20}{s^2+19s+90}\right|$$

$$\lim_{\omega\to\infty}\left\|\left|W_p(j\omega)S(j\omega)\right|+\left|W_m(j\omega)T(j\omega)\right|\right\| = \frac{1}{2} < 1$$

Hence, in terms of condition(8.80),closed-loop control system in example8.11 meets robust performance specifications.

8.10 Robustness of LQR and Kalman Filter Feedback Loops

8.10.1 Linear quadratic regulator(LQR) Feedback loops

Now consider the state space model which have been narrated:

$$\dot{x} = Ax + Bu \tag{8.87}$$

The state feedback law is as follows:

$$u(t) = -Kx(t) \tag{8.88}$$

Here, K is the optimal state feedback gain matrix. Taking Laplace transformations of equation(8.87) and (8.88) with zero initial conditions,

$$U(s) = -KX(s) \tag{8.89}$$

$$X(s) = (sE - A)^{-1} BU(s) = \phi(s)BU(s) \tag{8.90}$$

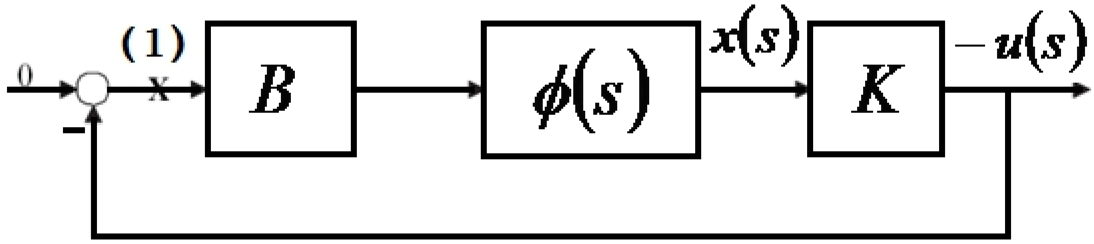


Figure8.20 A linear quadratic regulator feedback loop

$$\phi(s) = (sE - A)^{-1} \tag{8.91}$$

Relationships(8.89) and (8.90) can be represented as an LQR feedback loop shown in figure8.20. The LQR loop transfer matrix $G_{LQ}(s)$ is given by

$$G_{LQ}(s) = K\phi(s)B \tag{8.92}$$

Breaking the loop at (1) in figure8.20, return difference matrix:

$$E + G_{LQ}(s) = E + K\phi(s)B \tag{8.93}$$

Return difference equality on the basis of equation(8.87).

$$[E + G_{LQ}(-s)]^T R[E + G_{LQ}(s)] = R + [N\phi(-s)B]^T [N\phi(s)B] \tag{8.94a}$$

Here,

$$Q = N^T N \tag{8.94b}$$

Proof

Consider algebraic Riccati equation:

$$-PA - A^T P + PBR^{-1}B^T P = Q \tag{8.95}$$

Equation(8.95) may be written as

$$P(sE-A)+(-sE-A^T)P+PBR^{-1}B^TP=Q \tag{8.96a}$$

For

$$K=R^{-1}B^TP \tag{8.96b}$$

then we have

$$P(sE-A)+(-sE-A^T)P+K^TRK=Q \tag{8.97}$$

Equation(8.91) is replaced into above equation, and then we can get

$$P\phi^{-1}(s)+[\phi^T(-s)]^{-1}P+K^TRK=Q \tag{8.98}$$

Left-multiplying equation(8.98) by $B^T\phi^T(-s)$, and right-multiplying equation(8.98) by $\phi(s)B$,

$$\begin{aligned}&B^T\phi^T(-s)P\phi^{-1}(s)\phi(s)B+B^T\phi^T(-s)[\phi^T(-s)]^{-1}P\phi(s)B+B^T\phi^T(-s)K^TRK\phi(s)B\\&=B^T\phi^T(-s)Q\phi(s)B\end{aligned}$$

Namely,

$$B^T\phi^T(-s)PB+B^TP\phi(s)B+B^T\phi^T(-s)K^TRK\phi(s)B=B^T\phi^T(-s)Q\phi(s)B \tag{8.99}$$

From equation(8.96a),

$$RK=B^TP \tag{8.100}$$

Substituting equation(8.100) into equation(8.99), then we can achieve

$$B^T\phi^T(-s)K^TR+RK\phi(s)B+B^T\phi^T(-s)K^TRK\phi(s)B=B^T\phi^T(-s)Q\phi(s)B \tag{8.101}$$

Adding matrix R to both sides of equation(8.101), using equation(8.92) and equation(8.94b):

$$R+G_{LQ}^T(-s)R+RG_{LQ}(s)+G_{LQ}^T(-s)RG_{LQ}(s)=R+B^T\phi^T(-s)NN^T\phi(s)B \tag{8.102}$$

Equation(8.102) can be written as

$$[E+G_{LQ}(-s)]^TR[E+G_{LQ}(s)]=R+[N\phi(-s)B]^T[N\phi(s)B] \tag{8.103}$$

Proof is completed.

8.10.2 Gain and phase margin of a single-input system

For a single-input system, function $G_{LQ}(s)$ and R are scalar. Let

$$R=\rho>0 \tag{8.104}$$

Equation(8.103) reduces to

$$[E+G_{LQ}(-s)]^T[E+G_{LQ}(s)]=1+\frac{1}{\rho}[N\phi(-s)B]^T[N\phi(s)B] \tag{8.105}$$

Order $s=j\omega$ and substitute it into equation(8.105),

$$|1+G_{LQ}(j\omega)|^2=1+\frac{1}{\rho}\|N\phi(s)B\|^2 \tag{8.106}$$

Therefore,

$$\left|1+G_{LQ}(j\omega)\right| \geq 1 \tag{8.107}$$

From equation(8.107), it is known that the Nyquist plot of function $G_{LQ}(s)$ will be outside the circle of unit radius which center is at $-1+j0$.

Assume that l_1 is the factor by which the magnitude of input can change. In this case, the loop transfer function will be $l_1 G_{LQ}(s)$. Because the Nyquist plot of $G_{LQ}(s)$ will be outside the circle of unit radius which center is at $-1+j0$, it will only intersect the negative real axis beyond the point $G(-2,\ 0)$. Hence, the number of encirclements of $-1+j0$ will remain unchanged provided

$$l_1 > \frac{1}{2} \tag{8.108}$$

Therefore, the gain margin of the single-input LQR feedback loop shown in figure8.20 is at least between 0.5 and ∞.

Assume that ϕ_1 is the angle by which the phase of the input can change. In this case, the loop transfer function will be $e^{j\phi_1} G_{LQ}(s)$. Because the Nyquist plot of $G_{LQ}(s)$ will be outside the circle of unit radius which center is at $-1+j0$ and unit circles centered at $(0,\ 0)$ and $(-1,\ 0)$ intersect each other at $P_1\left(-1/2,\ \sqrt{3}/2\right)$ and $P_2\left(-1/2,\ -\sqrt{3}/2\right)$. The number of encirclements of $(-1,\ 0)$ will remain unchanged provided

$$|\phi_1| < \frac{\pi}{3} \tag{8.109}$$

Therefore, the phase margin of the single-input LQR feedback loop in figure8.20 is at least $\pi/3$ or 60^o.

8.10.3 Gain and phase margin of a multiple-input system

Assume that

$$R = \rho E \tag{8.110}$$

Then,from equation(8.94a),

$$\left[E+G_{LQ}(-s)\right]^T\left[E+G_{LQ}(s)\right] = E+\frac{1}{\rho}\left[N\phi(-s)B\right]^T\left[N\phi(s)B\right] \tag{8.111}$$

Similarly, substituting $s=j\omega$ into equation(8.111), and then we have

$$\left[E+G_{LQ}(-j\omega)\right]^T\left[E+G_{LQ}(j\omega)\right] = E+\frac{1}{\rho}\left[N\phi(-j\omega)B\right]^T\left[N\phi(j\omega)B\right] \tag{8.112}$$

Therefore,

$$\left[E+G_{LQ}(-j\omega)\right]^T\left[E+G_{LQ}(j\omega)\right] > E \tag{8.113}$$

Consider the control system in the block diagram of figure8.21. If $W^H + W > E$,

$E+G_{LQ}(j\omega)W(j\omega)$ is non-singular where W^H is complex conjugate transpose of matrix W,

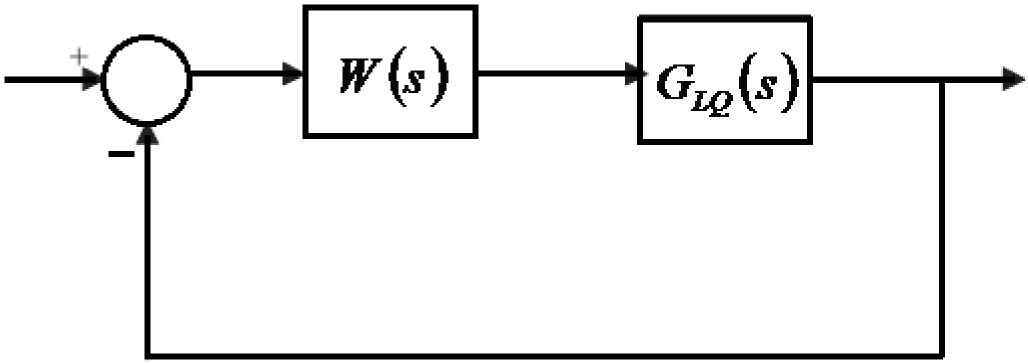


Figure8.21 Linear quadratic loop with an uncertainty $W(s)$

Proof

Assume $E+G_{LQ}(s)$ be singular. In this case, there will be nontrivial vector q such that

$$\left[E+G_{LQ}(j\omega)\right]q=0 \tag{8.114}$$

Or,

$$G_{LQ}Wq=-q \tag{8.115}$$

From equation(8.113), gain

$$G_{LQ}^H+G_{LQ}+G_{LQ}^H G_{LQ}\geq 0 \tag{8.116}$$

Here, $G_{LQ}^H(j\omega)$ is the complex conjugate transpose of $G_{LQ}(j\omega)$. Premultiplying and postmultiplying(8.116) by $q^H W^H$ and Wq, respectively,

$$q^H W^H G_{LQ}^H Wq+q^H W^H G_{LQ}Wq+q^H W^H G_{LQ}^H G_{LQ}Wq\geq 0 \tag{8.117}$$

Using equation(8.115), there is

$$-q^H Wq-q^H W^H q+q^H q\geq 0 \tag{8.118}$$

Or,

$$q^H\left(W+W^H-E\right)q\leq 0 \tag{8.119}$$

Inequality(8.119) contradicts the assumption $W^H+W>E$ in the statement of figure8.21.

It should be noted that the Nyquist condition for the stability of the system in figure8.21 is that $E+G_{LQ}(j\omega)W(j\omega)$ is non-singular. Therefore, the system in figure8.21 is stable when $W^H+W>E$.

When control matrix W is a diagonal matrix which diagonal elements are respectively $l_1,\ l_2,\ \cdots,\ l_m$. Then stable condition W^H+W will reduce to

$$l_i^H+l_i>1,\qquad i=1,2,\cdots,m \tag{8.120}$$

Condition(8.120) yields guaranteed levels of gain margin and phase margin of LQR

feedback loop which is shown in figure8.22.

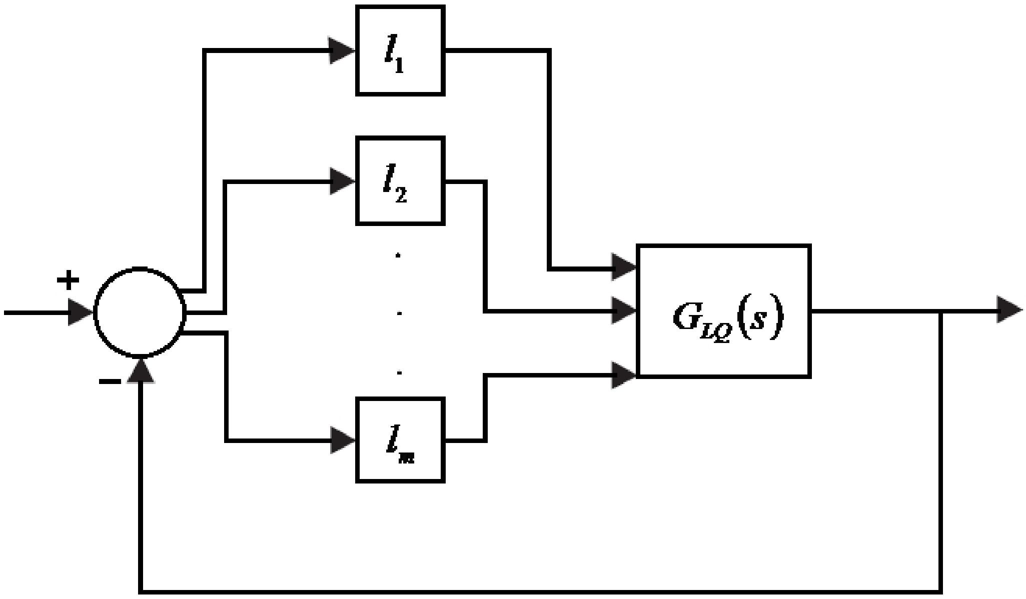


Figure8.22 A multi-input/multi-output linear quadratic regulator loop with a magnitude and uncertainty in each input channel

1. Gain margin

Assume that diagonal elements l_1, l_2, $\cdots$, l_m in matrix W are real. Then condition(8.22) yield

$$l_i > \frac{1}{2}, \qquad i = 1,2,\cdots,m \tag{8.121}$$

The LQR feedback loop in figure8.22 has a guaranteed infinite upward gain margin, and $1/2$ downward gain margin in each input channel independently and simultaneously.

2. Phase margin

Assume

$$l_i = e^{j\phi_i} \tag{8.122}$$

Then, condition(8.120) yields to

$$e^{-j\phi_i} + e^{j\phi_i} = 2\cos\phi_i > 1 \tag{8.123}$$

Hence,

$$|\phi_i| < \frac{\pi}{3} \tag{8.124}$$

The LQR feedback loop in figure8.23 has a guaranteed $\pm 60^o$ phase margin in each input channel independently and simultaneously.

Here, it is noted that the stability is not guaranteed if both a phase shift within $\pm 60^o$ and a gain change in the interval $(1/2,\ \infty)$ are simultaneously introduced in an input channel.

8.10.4 Kalman filter feedback loop

The Kalman filter deals with optimal selection of observer poles, and is dual to the linear quadratic control.

1.Kalman filter

(1) State equation(dynamics)

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \tag{8.125}$$

The vector $w(t)$ is a stochastic process called process noise. It is assumed that $w(t)$ is a continuous-time Gaussian white noise vector. Its mathematical characterization is

$$E(w(t)) = 0 \tag{8.126}$$

And

$$E(w(t))w^T(t+\tau) = w\delta(\tau) \tag{8.127}$$

The matrix w is called the intensity matrix with the property $w = w^T > 0$.

(2) Measurement equation

$$y = Cx(t) + \theta(t) \tag{8.128}$$

Here, $y(t)$ is the sensor noise vector and $\theta(t)$ is a continuous-time Gaussian white noise vector. It is assumed that

$$E[\theta(t)] = 0 \tag{8.129}$$

And

$$E(\theta(t))\theta^T(t+\tau) = \theta\delta(\tau) \tag{8.130}$$

Here, $\theta = \theta^T > 0$.

It is also assumed that process and measurement noise vectors are not correlated, i.e.,

$$E[\theta(t)w^T(t+\tau)] = E[\theta(t)]E[w^T(t+\tau)] = 0 \tag{8.131}$$

(3) Observer

The dynamics of state observers is the following:

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + L\left[y(t) - \hat{y}(t)\right]; \quad \hat{x}(0) = \hat{x}_0 \tag{8.132}$$

Here, L is an observer gain matrix.

Define the state error as

$$\tilde{x} = x(t) - \hat{x}(t) \tag{8.133}$$

Subtracting equation(8.132) from equation(8.125)

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + (w(t) - L\theta(t)) \tag{8.134}$$

Order

$$\psi(t) = w(t) - L\theta(t) \tag{8.135}$$

Then

$$E[\psi(t)] = E[w(t)] - LE[\theta(t)] = 0 \tag{8.136}$$

Consider

$$\begin{aligned}\psi(t)\psi^T(t+\tau) &= [w(t) - L\theta(t)][w(t+\tau) - L\theta(t+\tau)]^T \\ &= w(t)w^T(t+\tau) - L\theta(t)w^T(t+\tau) - w(t)\theta^T(t+\tau)L^T + L\theta(t)\theta^T(t+\tau)L^T\end{aligned} \tag{8.137}$$

Using equation(8.127), equation(8.131) and equation(8.132),

$$E[\psi(t)\psi^T(t+\tau)] = [w + L\theta(t)L^T(t+\tau)]\delta(\tau) \tag{8.138}$$

The state error dynamics in equation(8.134) is represented by a linear system subjected to white noise vector $\psi(t)$ with the intensity described by equation(8.138). Assuming that the matrix $(A - LC)$ is stable,

$$E\left[\tilde{x}(t)\right] \to 0, \quad as \quad t \to 0 \tag{8.139}$$

And

$$(A - LC)\sum + \sum(A - LC)^T + w + L\theta L^T = 0 \tag{8.140}$$

Here, $\sum$ is the steady state error covariance matrix defined as

$$\sum = \lim_{t\to\infty} E\left[\tilde{x}(t)\tilde{x}^T(t)\right] \tag{8.141}$$

Example8.12 Please find L such that

$$J = tr\left[\sum\right] = E\left(\tilde{x}_1^2(t)\right) + E\left(\tilde{x}_2^2(t)\right) + \cdots + E\left(\tilde{x}_n^2(t)\right) \tag{8.142}$$

is minimized subject to constraints(8.140).

Solution

Introducing symmetric matrix P as the Lagrange multiplier.

$$g\left(\sum, \ L, \ P\right) = tr\left(\sum\right) + tr\left(U\left(\sum, \ L\right)P\right) \tag{8.143}$$

Here, $U\left(\sum, \ L\right)$ is the left-side of equation(8.140). As a result,

$$g\left(\sum, \ L, \ P\right) = tr\left(\sum\right) + tr\left((A - LC)\sum P\right) + tr\left(\sum(A - LC)^T P\right) + tr(wP) + tr\left(L\theta L^T P\right) \tag{8.144}$$

After derivation of function $g\left(\sum, \ L, \ P\right)$ to variable $\sum$, gain

$$\frac{\partial g}{\partial \sum} = E + (A - LC)^T P + P(A - LC) = 0 \tag{8.145}$$

$$\frac{\partial g}{\partial L} = -P\sum C^T - P\sum C^T + PL\theta + PL\theta = 2P\left(-\sum C^T + L\theta\right) = 0 \tag{8.146}$$

$$\frac{\partial g}{\partial P} = \sum(A - LC)^T + (A - LC)\sum + w + L\theta L^T = 0 \tag{8.147}$$

Assuming that $(A - LC)$ is stable, $P > 0$ from the well-known results of the solution of Lyapunov stability criterion. Hence, from equation(8.146),

$$-\sum C^T + L\theta = 0 \tag{8.148}$$

Or,

$$L = \sum C^T \theta^{-1} \tag{8.149}$$

Substituting equation(8.149) into equation(8.140),

$$\sum A^T + A\sum \; - \sum C^T \theta^{-1} C \sum \; + w = 0 \tag{8.150}$$

Equation(8.150) is called the filter algebraic Riccati equation(FARE).

The important statistical properties of the Kalman filter are narrated as follows:

1. The state error vector is orthogonal to estimate the state vector in the following sense:

$$E\left(\tilde{x}\hat{x}^T\right) = 0 \tag{8.151}$$

Namely, the co-variance between state error variable $\tilde{x}$ and state estimation variable $\hat{x}$ is zero.

2. The following signal is white noise with zero mean,

$$v(t) = y(t) - C\hat{x}(t) \tag{8.152}$$

This signal is called the innovation process. Furthermore,

$$E\left(v(t)v(t+\tau)\right) = \theta\delta(t) \tag{8.153}$$

2.Kalman filter feedback loop

As we know, a Kalman filter is represented by

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + L\left(y - \hat{y}\right) \tag{8.154}$$

Here,

$$\hat{y} = C\hat{x}(t) \tag{8.155}$$

Taking the Laplace transformation of equation(8.154) with zero initial conditions,

$$\hat{x}(s) = \phi(s)Bu(t) + \phi(s)L\left(y(s) - \hat{y}(s)\right) \tag{8.156}$$

Here, $\phi(s) = (sE - A)^{-1}$, equation(8.91).

The relations between equation(8.155) and equation(8.156) are depicted in figure8.23. The Kalman filter feedback loop transfer matrix $G_{KF}(s)$ is given as

$$G_{KF}(s) = C\phi(s)L \tag{8.157}$$

Similar to equation(8.94a), it can be proved that

$$[E + G_{KF}(s)]\theta[E + G_{KF}(-s)]^T = \theta + \left(C\phi(s)N_f\right)\left(C\phi(-s)N_f\right)^T \tag{8.158}$$

Here,

$$W = N_f N_f^T \tag{8.159}$$

Assume that

$$\theta = \chi E ; \quad \chi > 0 \tag{8.160}$$

From equation(8.158) to equation(8.160),

$$[E + G_{KF}(s)][E + G_{KF}(-s)]^T = E + \frac{1}{\chi}(C\phi(s)N_f)(C\phi(-s)N_f)^T \tag{8.161}$$

Substituting $s = j\omega$ into equation(8.161).

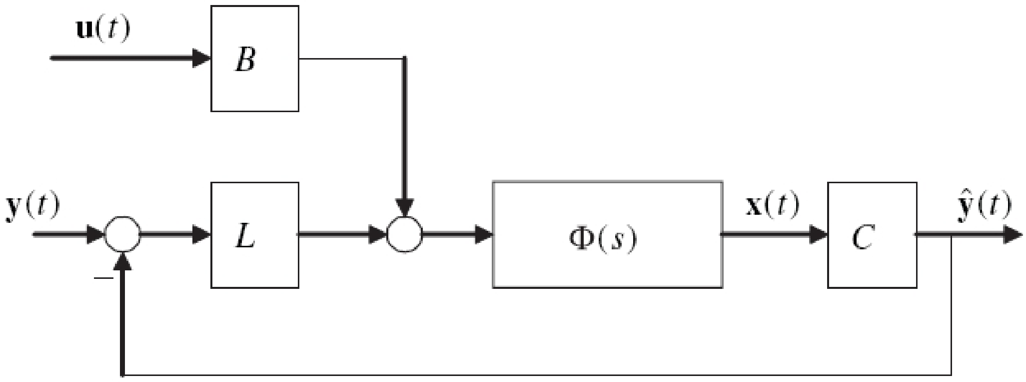


Figure8.23 A Kalman filter feedback loop

$$[E + G_{KF}(j\omega)][E + G_{KF}(-j\omega)]^T = E + \frac{1}{\chi}(C\phi(j\omega)N_f)(C\phi(-j\omega)N_f)^T \tag{8.162}$$

Since the second terms of right-side in equation(8.162) is positive semidefinite,

$$[E + G_{KF}(j\omega)][E + G_{KF}(-j\omega)]^T \geq E \tag{8.163}$$

Or,

$$\underline{\sigma}(E + G_{KF}(j\omega)) = \left[\left\|(E + G_{KF}(j\omega))^{-1}\right\|_2\right]^{-1} \geq 1 \ \text{ if } \ [E + G_{KF}]^{-1} \ \text{ exists} \tag{8.164}$$

$\underline{\sigma}(E + G_{KF}(j\omega))$ is minimum singular value.

Because the sensitivity of the Kalman filter(KF) loop is

$$S_{KF} = (E + G_{KF})^{-1} \tag{8.165}$$

Equation(8.164) yields

$$S_{KF}(j\omega)S_{KF}^T(-j\omega) \leq 1; \quad \forall \omega \tag{8.166}$$

Or,

$$\bar{\sigma}(S_{KF}(j\omega)) = \max_{\omega \neq 0}\left\|(S_{KF}(j\omega))\right\|_2 \leq 1 \tag{8.167}$$

Such property indicates that the Kalman filter loop will never amplify output disturbances at any frequency.

The complementary sensitivity function, T_{KF}, is given by

$$T_{KF} = 1 - S_{KF} \tag{8.168}$$

From equation(8.167) and equation(8.168) and properties of singular values,

$$\bar{\sigma}\left(T_{KF}\left(j\omega\right)\right)=\max_{\omega\neq 0}\left\|\left(T_{KF}\left(j\omega\right)\right)\right\|_2\leq 2;\quad \forall\omega \tag{8.169}$$

The stability of the KF loop is guaranteed to multiplicative modeling uncertainty provided

$$\bar{\sigma}\left(\Delta\left(j\omega\right)\right)\leq 0.5 \tag{8.170}$$

Here, $\Delta(s)$ relates the actual plant transfer function $G_\Delta(s)$ and model transfer matrix as follows:

$$G_\Delta(s)=\left(E+\Delta(s)\right)G(s) \tag{8.171}$$

Example8.13 Assume a Kalman filter loop for a simple mass, and the Kalman filter feedback loop transfer matrix $G_{KF}(s)$ is given by

$$G_{KF}=\frac{4.4721s+10}{s^2}$$

Therefore,

$$S_{KF}=\left(E+G_{KF}\right)^{-1}=\frac{s^2}{s^2+4.4721s+10},\quad T_{KF}=1-S_{KF}=\frac{4.4721s+10}{s^2+4.4721s+10}$$

$$\bar{\sigma}\left(S_{KF}\left(j\omega\right)\right)=\max_{\omega\neq 0}\left\|\left(S_{KF}\left(j\omega\right)\right)\right\|_2=\max_{\omega\neq 0}\left\|\frac{s^2}{s^2+\left(4.4721s\right)+10}\right\|_2$$

$$=\max_{\omega\neq 0}\sqrt{\frac{\omega^2}{\left(10-\omega^2\right)^2+\left(4.4721\omega\right)^2}}\leq\max_{\omega\neq 0}\sqrt{\frac{\omega^2}{\left(10\right)^2+\left(4.4721\omega\right)^2}}\leq\max_{\omega\neq 0}\sqrt{\frac{\omega^2}{\left(4.4721\omega\right)^2}}=\frac{1}{4.472}<1$$

$$\bar{\sigma}\left(T_{KF}\left(j\omega\right)\right)=\max_{\omega\neq 0}\left\|\left(T_{KF}\left(j\omega\right)\right)\right\|_2=\max_{\omega\neq 0}\left\|\left(1-S_{KF}\left(j\omega\right)\right)\right\|_2\leq 2$$

Hence both sensitivity and complementary sensitivity functions could satisfy the given conditions of equation(8.167) and equation(8.169).

8.11 Kharitonov's Theorem and Related Results

This section is mainly to focus on the seminal theorem of Kharitonov, which has motivated a variety of powerful results for general robustness problem.

8.11.1 Kharitonov's theorem on robust control stability

Khavritonov's theorem narrates robust stability of interval polynomials with lumped uncertainty and fixed degree of the form

$$p(s,\quad a)=\sum_{i=0}^{n}a_i s^i \tag{8.172}$$

where

$$a_i\in\left[a_i^-,\quad a_i^+\right],\quad i=0,1,\cdots\cdots,n$$

where $\left[a_i^-, \quad a_i^+\right]$ denotes the a known bounding interval for the *ith* component of uncertainty(the *ith* uncertain coefficient) a_i.

To describe Kharitonov's theorem for robust stability, it is first necessary to define four fixed polynomials associated with a family of interval polynomials.

Kharitonov's Polynomials

The four fixed Kharitonov's polynomials are associated with the intervals polynomial $p(s, \quad a)$.

$$\begin{aligned}
p_1(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + a_5^- s^5 + a_6^+ s^6 + \cdots \\
p_2(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + a_5^+ s^5 + a_6^- s^6 + \cdots \\
p_3(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + a_5^- s^5 + a_6^- s^6 + \cdots \\
p_4(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4 + a_5^+ s^5 + a_6^+ s^6 + \cdots
\end{aligned} \tag{8.173}$$

An example is provided to demonstrate above Kharitonov polynomials constructed by inspection in the following.

Example8.15 Consider the following interval polynomial with fixed degree6:

$$p(s, \quad a) = [2, \quad 3]s^6 + [1, \quad 8]s^5 + [3, \quad 12]s^4 + [5, \quad 6]s^3 + [4, \quad 7]s^2 + [9, \quad 11]s + [6, \quad 15]$$

Please derive the four Kharitonov polynomials.

Solution

According to equation(8.173), the four Kharitonov polynomials can be written into the following form:

$$p_1(s) = 6 + 9s + 7s^2 + 6s^3 + 3s^4 + s^5 + 3s^6$$

$$p_2(s) = 15 + 11s + 4s^2 + 5s^3 + 12s^4 + 8s^5 + 2s^6$$

$$p_3(s) = 15 + 9s + 4s^2 + 6s^3 + 12s^4 + s^5 + 2s^6$$

$$p_4(s) = 6 + 11s + 7s^2 + 5s^3 + 3s^4 + 8s^5 + 3s^6$$

Now, the Kharitonov's theorem will be presented.

Kharitonov's theorem1

An interval polynomial $p(s, \quad a)$ with invariant degree n is robustly stable if and only if its four associated Kharitonov polynomials are stable.

Example8.16 Consider the following interval polynomial with fixed degree5:

$$p(s, \quad a) = [1, \quad 3]s^5 + [3, \quad 6]s^4 + [4, \quad 7]s^3 + [5, \quad 9]s^2 + [3, \quad 4]s + [2, \quad 5]$$

Please judge whether the four Kharitonov polynomial is robustly stable.

Solution

According to equation(8.173), the four Kharitonov polynomials can be written into the following form:

$$\begin{aligned}
p_1(s) &= 2 + 3s + 9s^2 + 7s^3 + 3s^4 + s^5 \\
p_2(s) &= 5 + 4s + 5s^2 + 4s^3 + 6s^4 + 3s^5 \\
p_3(s) &= 5 + 3s + 5s^2 + 7s^3 + 6s^4 + s^5 \\
p_4(s) &= 2 + 4s + 9s^2 + 4s^3 + 3s^4 + 3s^5
\end{aligned}$$

According to Kharitonov theorem1, in order to guaranteed the robust stability of the given interval polynomial, it is sufficient to test the stability of above four Kharitonov polynomials. Algebraic stability criterion can be employed to accomplish the judgment of the stability of above four Kharitonov polynomials. After Roth's stable criterion is applied to above four Kharitonov polynomials, we can respectively get the following four Routh's table.

1. Routh's table of polynomial $p_1(s)$

s^5	1	7	3
s^4	3	9	2
s^3	4	$-7/3$	0
s^2	$29/4$	2	0
s^1	$107/77$	0	0
s^0	2	0	0

2. Routh's table of polynomial $p_2(s)$

s^5	3	4	4
s^4	6	5	5
s^3	$3/2$	$3/2$	0
s^2	-1	5	0
s^1	$18/2$	0	0
s^0	5	0	0

3. Routh's table of polynomial $p_3(s)$

s^5	1	7	3
s^4	6	5	5
s^3	$37/6$	$13/6$	0
s^2	$107/37$	5	0
s^1	$-909/107$	0	0
s^0	5	0	0

4. Routh's table of polynomial $p_4(s)$

s^5	3	4	4
s^4	3	9	2
s^3	-5	2	0
s^2	$51/5$	2	0
s^1	$152/51$	0	0
s^0	2	0	0

Among above Routh's table, it is easily known that polynomials $p_2(s),\quad p_3(s),\quad p_4(s)$ are unstable. Hence the interval polynomial $p(s,\quad a)$ is not robustly stable.

Kharitonov's test for robust stability can be significantly simplified if the interval polynomial is of degree 5, 4, and 3. In these cases, the Kharitonov polynomials for performing the test are 3, 2, or 1 in number, respectively, as against the four polynomials of the general case. More precisely, we have the following propositions.

1) Propositions 1

An interval polynomial $p(s,\ a)$ with invariant degree 5 is robustly stable if and only if the Kharitonov polynomials $p_1(s)$, $p_2(s)$, and $p_3(s)$ are stable.

2) Propositions 2

An interval polynomial $p(s,\ a)$ with invariant degree 4 is robustly stable if and only if the Kharitonov polynomials $p_2(s)$, and $p_3(s)$ are stable.

3) Proposition 3

An interval polynomial $p(s,\ a)$ with invariant degree 3 is robustly stable if and only if the Kharitonov polynomials $p_3(s)$ are stable.

Although Kharitonov's theorem is a very important result in the area of robustness analysis, it has several limitations. The most important limitations are as follows:

(a) Kharitonov's theorem is applicable only in problems for which the stability region is the open left-half plane. In other words, Kharitonv's theorem can not be applied in the case of discrete-time systems.

(b) Kharitonov's theorem is applicable only in the case of interval polynomials whose coefficients vary independently. In the more general case of interval polynomials of the following form

$$p(s,\ a) = \sum_{i=0}^{n} f_i(a_1,\ a_2,\ \cdots,\ a_m)s^i;\quad i = 1,2,\cdots,n \tag{8.174}$$

where $f_i(a_1,\ a_2,\ \cdots,\ a_m)$, $i = 1,2,\cdots,n$ are multilinear functions of the uncertain coefficients $a_1,\ a_2,\ \cdots,\ a_m$, Kharitonov's theorem fails to give an answer to the question of the robust stability of interval polynomials of the form(8.174). The interested readers can refer to other robust materials.

8.11.2 The sixteen-plant theorem

The robust stability has been discussed in the previous section. Here, the analytical results presented in the subsection8.11.1 needs to be generalized in order to develop a technique for the design of robustly stabilizing compensators. In particular, the form of proper first-order compensators is the following:

$$F(s) = \frac{K(s-z)}{s-p} \tag{8.175}$$

Equation(8.175) robustly stabilizes a strictly proper interval plant family of the following plant:

$$G(s,\quad a,\quad b)=\frac{A(s,\quad a)}{B(s,\quad b)}\frac{\sum_{j=0}^{m}a_j s^j}{s^n+\sum_{i=0}^{n}a_i s^i}\quad m<n \tag{8.176}$$

$$a_j\in\left[a_j^-,\quad a_j^+\right],\quad j=0,1,\cdots,m,\quad b_i\in\left[b_i^-,\quad b_i^+\right],\quad i=0,1,\cdots,n-1$$

Here $\left[a_j^-,\quad a_j^+\right]$ and $\left[b_i^-,\quad b_i^+\right]$ denote a priori known bounding intervals for the jth and the ith components of uncertainties a_j and b_i, respectively.

The compensator of the form(8.175) robustly stabilizes the interval plant family(8.176), if for all $a_j\in\left[a_j^-,\quad a_j^+\right],\quad j=0,1,\cdots,m$ and $b_i\in\left[b_i^-,\quad b_i^+\right],\quad i=0,1,\cdots,n-1$, the resulting closed-loop polynomial has Hurwitz stability(its roots lie in the open left-half plane). Function $F(s)$ is called a robust stabilizer of the interval plant family(8.176).

$$p_c(s,\quad a,\quad b)=K(s-z)A(s,\quad a)+(s-p)B(s,\quad b) \tag{8.177}$$

The interval plant family given in equation(8.176) wit compensator interconnected is shown in figure8.24.

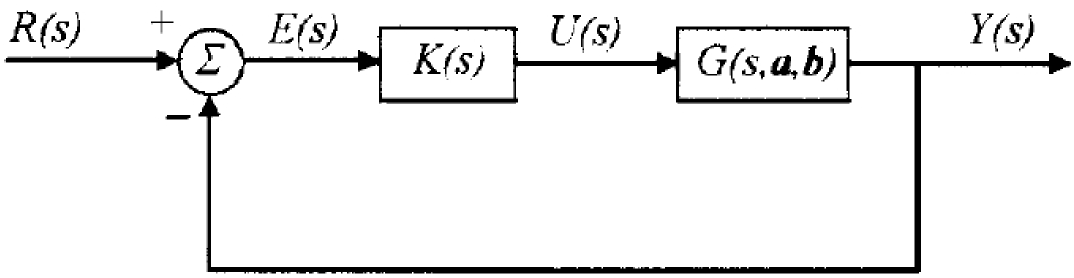


Figure8.24 Interval plant family interconnected with a first-order compensator

For the numerator of equation(8.176), the following four Kharitonov polynomials can be gained:

$$A_1(s)=a_0^-+a_1^-s+a_2^+s^2+a_3^+s^3+a_4^-s^4+a_5^-s^5+a_6^+s^6+\cdots$$
$$A_2(s)=a_0^++a_1^+s+a_2^-s^2+a_3^-s^3+a_4^+s^4+a_5^+s^5+a_6^-s^6+\cdots$$
$$A_3(s)=a_0^++a_1^-s+a_2^-s^2+a_3^+s^3+a_4^+s^4+a_5^-s^5+a_6^-s^6+\cdots$$
$$A_4(s)=a_0^-+a_1^+s+a_2^+s^2+a_3^-s^3+a_4^-s^4+a_5^+s^5+a_6^+s^6+\cdots$$

For the denominator of equation(8.176), the following four Kharitonov polynomials can be gained:

$$B_1(s)=b_0^-+b_1^-s+b_2^+s^2+b_3^+s^3+b_4^-s^4+b_5^-s^5+b_6^+s^6+\cdots$$
$$B_2(s)=b_0^++b_1^+s+b_2^-s^2+b_3^-s^3+b_4^+s^4+b_5^+s^5+b_6^-s^6+\cdots$$
$$B_3(s)=b_0^++b_1^-s+b_2^-s^2+b_3^+s^3+b_4^+s^4+b_5^-s^5+b_6^-s^6+\cdots$$
$$B_4(s)=b_0^-+b_1^+s+b_2^+s^2+b_3^-s^3+b_4^-s^4+b_5^+s^5+b_6^+s^6+\cdots$$

Considering all combinations of the $A_i(s),\quad i=1,2,3,4$, and $B_k(s),\quad k=1,2,3,4$, we can get 16 Kharitonov plants for $i,k=1,2,3,4$:

$$G_{ik}(s)=\frac{A_i(s)}{B_k(s)} \tag{8.178}$$

For these extreme plants, when it is said that $F(s)$ stabilizes $G_{ik}(s)$, the following closed-loop polynomial is asymptotically stable.

$$p_{c,ik}(s)=K(s-z)A_i(s)+(s-p)B_k(s) \tag{8.179}$$

Here, we can narrate the so-called sixteen plant theorem.

Sixteen plant theorem

A proper first-order compensator of the form(8.175) robustly stabilizes the interval plant family(8.176), if and only if it stabilizes all of the 16 Kharitonov plants $G_{ik}(s)$, $i,k=1,2,3,4$.

Example8.17 Consider the following interval family:

$$G(s,\quad a,\quad b)=\frac{[2,\quad 5]s^2+[6,\quad 9]s+[3,\quad 11]}{s^3+[4,\quad 6]s^2+[1,\quad 8]s+[5,\quad 7]}$$

Please determine one proper first-order compensator that can robustly stabilize the family and one can not.

Solution

According to equation(8.178), the 16 Kharitonov plants associated with this family are:

$$G_{11}(s)=\frac{2s^2+6s+11}{s^3+4s^2+s+7},\quad G_{12}(s)=\frac{2s^2+6s+11}{s^3+6s^2+8s+5}$$

$$G_{13}(s)=\frac{2s^2+6s+11}{s^3+6s^2+s+5},\quad G_{14}(s)=\frac{2s^2+6s+11}{s^3+4s^2+8s+7}$$

$$G_{21}(s)=\frac{5s^2+9s+3}{s^3+4s^2+s+7},\quad G_{22}(s)=\frac{5s^2+9s+3}{s^3+6s^2+8s+5}$$

$$G_{23}(s)=\frac{5s^2+9s+3}{s^3+6s^2+s+5},\quad G_{24}(s)=\frac{5s^2+9s+3}{s^3+4s^2+8s+7}$$

$$G_{31}(s)=\frac{5s^2+6s+3}{s^3+4s^2+s+7},\quad G_{32}(s)=\frac{5s^2+6s+3}{s^3+6s^2+8s+5}$$

$$G_{33}(s)=\frac{5s^2+6s+3}{s^3+4s^2+8s+7},\quad G_{34}(s)=\frac{5s^2+6s+3}{s^3+6s^2+s+5}$$

$$G_{41}(s)=\frac{2s^2+9s+11}{s^3+4s^2+s+7},\quad G_{42}(s)=\frac{2s^2+9s+11}{s^3+6s^2+8s+5}$$

$$G_{43}(s)=\frac{2s^2+9s+11}{s^3+6s^2+s+5},\quad G_{44}(s)=\frac{2s^2+9s+11}{s^3+4s^2+8s+7}$$

With a particular first-order compensator, say $F(s)=(s-1)/(s+1)$, it can easily be verified that the closed-loop polynomial, associated with the Kharitonov plant $G_{32}(s)$, is given by

$$p_{c,11}(s) = (s-1)(2s^2+6s+11)+(s+1)(s^3+4s^2+s+7)$$
$$= s^4+7s^3+9s^2+13s-4$$
$$p_{c,32}(s) = (s-1)(5s^2+6s+3)+(s+1)(s^3+6s^2+8s+5)$$
$$= s^4+12s^3+15s^2+10s+2$$

Hence, controller $F(s)=(s-1)/(s+1)$ can not robustly stabilize the given plant family. Certainly, we easily testify that function $F(s)=1+\dfrac{1}{s}$ can robustly stabilize the given plant family.

The sixteen plant theorem is not only used to determine the robust stability of system, but also to design some parameters as a design tool.

Example8.18 Consider the following interval family:

$$G(s,\quad a,\quad b)=\frac{[1,\quad 1.5]s+[0.5,\quad 1]}{s^3+[2,\quad 3]s^2+[1,\quad 2]s+[3,\quad 4]}$$

Please determine one proper first-order compensator $F(s)=K_1+K_2/s$ that can robustly stabilize the interval plant family.

Solution

To solve this problem, we should firstly construct the 16 Kharitonov plants associated with the given interval plant family:

$$G_{11}(s)=\frac{s+0.5}{s^3+2s^2+s+4},\quad G_{12}(s)=\frac{s+0.5}{s^3+3s^2+s+3}$$
$$G_{13}(s)=\frac{s+0.5}{s^3+2s^2+2s+4},\quad G_{14}(s)=\frac{s+0.5}{s^3+3s^2+2s+3}$$
$$G_{21}(s)=\frac{1.5s+1}{s^3+2s^2+s+4},\quad G_{22}(s)=\frac{1.5s+1}{s^3+3s^2+s+3}$$
$$G_{23}(s)=\frac{1.5s+1}{s^3+2s^2+2s+4},\quad G_{24}(s)=\frac{1.5s+1}{s^3+3s^2+2s+3}$$
$$G_{31}(s)=\frac{s+1}{s^3+2s^2+s+4},\quad G_{32}(s)=\frac{s+1}{s^3+3s^2+s+3}$$
$$G_{33}(s)=\frac{s+1}{s^3+2s^2+2s+4},\quad G_{34}(s)=\frac{s+1}{s^3+3s^2+2s+3}$$
$$G_{41}(s)=\frac{1.5s+0.5}{s^3+2s^2+s+4},\quad G_{42}(s)=\frac{1.5s+0.5}{s^3+3s^2+s+3}$$
$$G_{43}(s)=\frac{1.5s+0.5}{s^3+2s^2+2s+4},\quad G_{44}(s)=\frac{1.5s+0.5}{s^3+3s^2+2s+3}$$

With the given form $F(s)=K_1+K_2/s$ of the PI controller, the closed-loop polynomials associated with 16 Kharitonov plants can be easily obtained. For example, for the Kharitonov plant G_{34}, the closed-loop polynomials can be found to be the following in terms of equation(8.179):

$$p_{c,34}(s)=(K_1 s+K_2)(s+1)+s(s^3+3s^2+2s+3)=s^4+3s^3+(2+K_1)s^2+(K_1+K_2+3)s+K_2$$

Roth's stability criterion can be employed to determine the range unknown parameters to design related system.

The Routh table can be written out as follows:

$$\begin{array}{cccc}
s^4 & 1 & 2+K_1 & K_2 \\
s^3 & 3 & K_1+K_2+3 & 0 \\
s^2 & \dfrac{2K_1-K_2+3}{3} & K_2 & 0 \\
s^1 & 3+K_1+K_2-\dfrac{9K_2}{2K_1-K_2+3} & 0 & 0 \\
s^0 & K_2 & 0 & 0
\end{array}$$

In similar manner, the remaining 15 Kharitonov plants may be applied by Roth table. In order to ensure that system is robustly stable, the first column of all Routh tables are both greater than zero. From above Routh table of closed-loop polynomial $p_{c,34}$, the following inequality can be gained:

$$2K_1>K_2-3,\ \ K_2>0 \ \text{ and } \ 2K_1^2-K_2^2+K_1K_2+9K_1-3K_2+9>0$$

When final range of parameters K_1 and K_2 is commonly ensured by the cross-section of the range of parameters from each Routh table, then the related parameter of PI controller may be selected. For example, a robust PI stabilizer can be given by

$$F(s)=15+\frac{10}{s}$$

Certainly, the sixteen plant theorem can be extended to the more general class of compensators;but it can not be applied into all class of compensators. Readers may result in related material of the sixteen plant theorem.

$$F(s)=\frac{K(s-z)}{s^q(s-p)},\quad q>1$$

Chapter summary

The PID control and robust control are very important topics in modern control theory. Firstly, the PID structure and control system with PID controller are narrated and explained, among which some specific examples are provided for readers to easily comprehend them. Furthermore, basic thoughts of robust control is provided for readers. Then on the topic of robust control, its basic model, stability and performance and LQR and Kalman filter are respectively explained and demonstrated. Finally, Kharitonov's theorem is utilized to show how to judge the stability of the control system with uncertain parameters. In this chapter, readers need to grasp the knowledge of PID controller and robust stability such as gain and phase margin of system and the sixteen-plant theorem, and so on.

Related Readings

[1]Robert L. Williams II and Douglas A. Lawrence. *"Linear State-Space Control Systems"*, Hoboken: John Wiley&Sons Ltd, 2007, USA.
[2]Tamer Mansour. *"PID Control,Implementation and Tuning"*, Trdine: InTech, 2011, Croatia.
[3]KarlJ. Astrom, Tore Hagglund. *"PID Controllers(second edition)"*, Research Triangle Park: Instrument Society of America,1995, USA.
[4]Antonio Visioli. *"Practical PID Control"*, London: Springer London limited, 2006, UK.
[5]Michael Green, David J. N. Limebeer. *"Linear Robust Control"*, New York: Dover Publications, 2012, USA.
[6]Mohammad A., Khosravi, Hamid D. Taghirad. "Robust PID control of fully-constrained cable driven parallel robots", Mechatronics, Volume 24, Issue 2, March 2014, Pages 87-97.
[7]Flavio Manenti, Francesco Rossi, Alexey G. Goryunov, et al. *"Fuzzy adaptive control system of a non-stationary plant with closed-loop passive identifier"*, Resource-Efficient Technologies, Volume 1, Issue 1, July 2015, Pages 10-18.
[8]Carlos E. de Souza, Daniel Coutinho. *"Robust stability and control of uncertain linear discrete-time periodic systems with time-delay"*, Automatica, Volume 50, Issue 2, February 2014, Pages 431-441.
[9]Julio E. Normey-Rico, Pedro Garcia, Antonio Gonzalez. *"Robust stability analysis of filtered Smith predictor for time-varying delay processes"*, Journal of Process Control, Volume 22, Issue 10, December 2012, Pages 1975-1984.
[10]Shabnam Rasoulian, Luis A. Ricardez-Sandoval. *"Robust multivariable estimation and control in an epitaxial thin film growth process under uncertainty"*, Journal of Process Control, Volume 34, October 2015, Pages 70-81.

Review Questions

8.1. Please explain the basic structure of PID controller.

8.2. Please give out the common characteristic equation of the unit negative feedback system with PID controller.

8.3. Do you think how to implement a digital PID controller?

8.4. What condition can lead-lag compensator be considered as a PID controller?

8.5. Please resumptively tell your comprehension on basic thoughts of robust control?

8.6. Please say out robust model types of linear uncertainty system.

8.7. Please explain the state equation and measurement equation of Kalman filtering.

8.8. Please talk about your comprehension on Kharitonov's theorem on robust control stability.

8.9. You think how to utilize the sixteen-plant theorem to judge the stability of robust control system.

8.10. Please talk about the gain margin and phase margin of SISO system and MIMO system.

Problems

Problem8.1. Assume a control system with PID controller is shown in figure8.4. The transfer function of plant component is the following:

$$G(s)=\frac{K}{s(1+sT_1)(1+sT_2)}$$

Using PID controller is to achieve arbitrary pole placement and to simultaneously drive the position steady-error to zero. Please give out the relative parameters K_p, K_I, K_D of PID controller.

Problem8.2.In figure8.5, the unit step response of an executor is given. Transient response parameters are $t_d = 100\sec$ and $t_r = 60\sec$. Please use table8.1 to determine the parameters K_p, T_i and T_d of the PID controller.

Problem8.3.Consider the uncertain feedback control system in figure8.13. Assume that the uncertain plant transfer function is given by the following expression:

$$H_p(s)=H(s)[1+W_m(s)\Delta_m(s)]$$

Where

$$H(s)=\frac{2}{s^2+2s+3} \quad \text{and} \quad W_m=\frac{3}{s}$$

While the controller $K(s)$ is a PID controller of the form $K(s)=s+2$. Please determine and judge whether the closed-loop system is robustly stable.

Problem8.4. Consider the following interval polynomial with fixed degree6:

$$p(s,\ a)=[1,\ \ 2]s^6+[2,\ \ 3]s^5+[3,\ \ 6]s^4+[4,\ \ 12]s^3+[2,\ \ 2]s^2+[5,\ \ 10]$$

Please derive the four Kharitonov polynomials.

Problem8.5. Consider the uncertain feedback control system in figure8.16 with

$$H(s)=\frac{2}{s^2+2s+5} \quad \text{and} \quad K(s)=s+1$$

The design performance specifications for the closed-loop control system be the following:

(1)Steady-state tracking error $A=0.1$

(2)Maximum peak magnitude $M=1$ of sensitivity function $S(s)$

(3)Minimum bandwidth $\omega_b = 0.1 rad/\sec$.

Please judge whether the performance specifications for the closed-loop control system meet the nominal performance requirement.